

PICTURES OF KK-THEORY FOR REAL C^* -ALGEBRAS

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ABSTRACT. We give a systematic account of the various pictures of KK-theory for real C^* -algebras, proving natural isomorphisms between the groups that arise from each picture. As part of this project, we develop the universal properties of KK-theory, and we use *CRT*-structures to prove that a natural transformation $F(A) \rightarrow G(A)$ between homotopy equivalent, stable, half-exact functors defined on real C^* -algebras is an isomorphism provided it is an isomorphism on the smaller class of C^* -algebras. Finally, we develop *E*-theory for real C^* -algebras and use that to obtain new negative results regarding the problem of approximating almost commuting real matrices by exactly commuting real matrices.

1. INTRODUCTION AND PRELIMINARIES

A real C^* -algebra is a Banach $*$ -algebra A over the real numbers such that $\|x^*x\| = \|x\|^2$ holds for all x and such that every element of the form $1 + x^*x$ is invertible in the unitization of A (see [32]). In this paper, we will adopt the term *R^* -algebra* instead. Every R^* -algebra is isometrically isomorphic to a closed $*$ -algebra of bounded operators on a Hilbert space over \mathbb{R} . In addition, every R^* -algebra is isomorphic to the $*$ -algebra of fixed elements of a C^* -algebra with a conjugate linear involution.

In Kasparov's seminal paper [21] introducing KK-theory, he simultaneously considered both R^* -algebras and C^* -algebras. Since then, many alternate but equivalent or closely related pictures of KK-theory have been introduced and developed by various authors ([9], [10], [15], [16], [28], [34]). The ability to move among the various pictures has contributed immensely to the utility of KK-theory as a tool for solving problems. However, these authors have not following Kasparov's lead and the alternate pictures of KK-theory have been developed primarily in the complex case.

In recent years, substantial progress has been made in developing the tools to study R^* -algebras including the development of united *K*-theory and the universal coefficient theorem in [2] and [3]. This has led to a classification of purely infinite simple R^* -algebras (in [5]) and the classification of real forms of UHF-algebras that are stable over the CAR-algebra (in [33]).

Given the centrality of KK-theory for these projects, there has been a need to develop a systematic account of the various pictures of KK-theory for R^* -algebras. In this paper, we will develop several of the alternate pictures of KK-theory in the context of R^* -algebras and prove the appropriate equivalent theorems. In particular, in this paper we will consider the following pictures of KK-theory and prove appropriate equivalence theorems for each: the standard Kasparov bimodule picture of KK-theory, the Fredholm picture (both in Section 2), the universal property picture (Section 3), the extension picture (Section 4), and suspended *E*-theory using asymptotic morphisms (Section 5).

Furthermore, we prove a more general theorem which reduces the work required to replicate many of these equivalent theorems and promises to ease the way for similar projects in the future. Suppose that $\mu: F \rightarrow G$ is a natural transformation between homotopy invariant, stable, half exact

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functors. We prove that if μ is an isomorphism for all C^* -algebras, then it is an isomorphism for R^* -algebras. This is accomplished in Section 3 when we develop the universal properties of KK-theory and K -theory for R^* -algebras.

In the last two sections, we will apply these ideas, using KK-theory to prove the existence of certain asymptotic morphisms, which in turn is used to obtain new results for the problem of approximating a set of almost commuting matrices over the field of real numbers. In particular, let the Halmos number be the largest integer d such that whenever d real self-adjoint matrices almost commute (pairwise) they can be approximated by d pairwise commuting matrices. More precisely, for all $\epsilon > 0$ there should be a $\delta > 0$ such that if $\{H_i\}_{i=1}^d$ is a collect of d self-adjoint matrices such that

$$\|H_r\| \leq 1 \quad \text{and} \quad \|[H_r, H_s]\| \leq \delta ,$$

for all r, s , then there exists a collection $\{K_i\}_{i=1}^d$ of self-adjoint matrices such that

$$\|K_r\| \leq 1 \quad \text{and} \quad \|[K_r, K_s]\| = 0 \quad \text{and} \quad \|H_r - K_r\| \leq \epsilon .$$

Furthermore, the dependence of δ on ϵ must be uniform, independent of the dimension of the matrices H_r . It is shown in [27] that in the context of real matrices, the statement is true for $d = 2$. We will show in Section 7 the statement is false for $d = 5$. Therefore, the Halmos number for real matrices is between 2 and 4, inclusive.

2. THE STANDARD AND FREDHOLM PICTURES OF KK-THEORY

We take the following definition from Section 2.3 of [32] to be the standard definition of KK-theory for R^* -algebras. It is essentially the same as that in [21] where it was simultaneously developed for both R^* -algebras and C^* -algebras.

Definition 2.1. Let A be a graded separable R^* -algebra and B be an R^* -algebra with σ -unital.

- (i) A Kasparov $(A-B)$ -bimodule is a triple (E, ϕ, T) where E is a countably generated graded Hilbert B -module, $\phi : A \rightarrow \mathcal{L}(E)$ is a graded $*$ -homomorphism, and T is an element of $\mathcal{L}(E)$ of degree 1 such that

$$(T - T^*)\phi(a), (T^2 - 1)\phi(a), \text{ and } [T, \phi(a)]$$

lie in $\mathcal{K}(E)$ for all $a \in A$.

- (ii) Two triples (E_i, ϕ_i, T_i) are *unitarily equivalent* if there is unitary U in $\mathcal{L}(E_0, E_1)$, of degree zero, intertwining the ϕ_i and T_i .
- (iii) If (E, ϕ, T) is a Kasparov $(A-B)$ -bimodule and $\beta : B \rightarrow B'$ is a $*$ -homomorphism of R^* -algebras, then the pushed-forward Kasparov $(A-B')$ -bimodule is defined by

$$\beta_*(E, \phi, T) = (E \hat{\otimes}_\beta B', \phi \hat{\otimes} 1, T \hat{\otimes} 1) .$$

- (iv) Two Kasparov $(A-B)$ -bimodules (E_i, ϕ_i, T_i) for $i = 0, 1$ are *homotopic* if there is a Kasparov bimodule $(A-IB)$, say (E, ϕ, T) , such that $(\varepsilon_i)_*(E, \phi, T)$ and (E_i, ϕ_i, T_i) are unitarily equivalent for $i = 0, 1$, where $IB = C([0, 1], B)$ and ε_i denotes the evaluation map.
- (v) A triple (E, ϕ, T) is *degenerate* if the elements

$$(T - T^*)\phi(a), (T^2 - 1)\phi(a), \text{ and } [T, \phi(a)]$$

are zero for all $a \in A$. By Proposition 2.3.3 of [32], degenerate bimodules are homotopic to trivial bimodules.

- (vi) $\text{KK}(A, B)$ is defined to be the set of homotopy equivalence classes of Kasparov $(A-B)$ -bimodules.

The following theorem summarizes the principal properties of KK-theory for R^* -algebras from Chapter 2 of [32].

Proposition 2.2. $\mathrm{KK}(A, B)$ is an abelian group for separable A and σ -unital B . As a functor on separable R^* -algebras (contravariant in the first argument and covariant in the second argument), it is homotopy invariant, stable, and has split exact sequences in both arguments. Furthermore, there is a natural associate pairing (the intersection product)

$$\otimes_{\mathrm{KK}}: \mathrm{KK}(A, C \otimes B) \otimes \mathrm{KK}(C \otimes A', B') \rightarrow \mathrm{KK}(A \otimes A', B \otimes B').$$

We now turn to the Fredholm picture of KK-theory, which was developed in [15] with only the situation of C^* -algebras in mind. However, the approach goes through the same for R^* -algebras, as follows.

Definition 2.3. Let A and B be separable R^* -algebras.

- (i) A triple (ϕ_+, ϕ_-, U) , where $\phi_{\pm} : A \rightarrow \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ are $*$ -homomorphisms, and U is an element of $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ such that

$$U\phi_+(a) - \phi_-(a)U, \phi_+(a)(U^*U - 1), \text{ and } \phi_-(a)(UU^* - 1)$$

lie in $\mathcal{K}_{\mathbb{R}} \otimes B$ for all $a \in A$ is called a $\mathrm{KK}(A, B)$ -cycle.

- (ii) Two $\mathrm{KK}(A, B)$ -cycles $(\phi_+^1, \phi_-^1, U^1)$ and $(\phi_+^2, \phi_-^2, U^2)$ are *homotopic* if there is a $\mathrm{KK}(A, IB)$ -cycle (ϕ_+, ϕ_-, U) such that $(\varepsilon_i \phi_+, \varepsilon_i \phi_-, \varepsilon_i(U)) = (\phi_+^i, \phi_-^i, U^i)$, where $\varepsilon_i : \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes IB) \rightarrow \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ is induced by evaluation at i .
- (iii) A $\mathrm{KK}(A, B)$ -cycle (ψ_+, ψ_-, V) is *degenerate* if the elements

$$V\psi_+(a) - \psi_-(a)V, \psi_+(a)(V^*V - 1), \text{ and } \psi_-(a)(VV^* - 1)$$

are zero for all $a \in A$.

- (iv) The sum $(\phi_+, \phi_-, U) \oplus (\psi_+, \psi_-, V)$ of two $\mathrm{KK}(A, B)$ -cycles is the $\mathrm{KK}(A, B)$ -cycle

$$\left(\begin{pmatrix} \phi_+ & 0 \\ 0 & \psi_+ \end{pmatrix}, \begin{pmatrix} \phi_- & 0 \\ 0 & \psi_- \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right)$$

where the algebra $M_2(\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B))$ is identified with $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ by means of some $*$ -isomorphism $M_2(\mathcal{K}_{\mathbb{R}}) \cong \mathcal{K}_{\mathbb{R}}$, which is unique up to homotopy by Section 1.17 of [21].

- (v) Two cycles $(\phi_+^0, \phi_-^0, U^0)$ and $(\phi_+^1, \phi_-^1, U^1)$ are said to be *equivalent* if there exist degenerate cycles $(\psi_+^0, \psi_-^0, V^0)$ and $(\psi_+^1, \psi_-^1, V^1)$ such that

$$(\phi_+^0, \phi_-^0, U^0) \oplus (\psi_+^0, \psi_-^0, V^0) \quad \text{and} \quad (\phi_+^1, \phi_-^1, U^1) \oplus (\psi_+^1, \psi_-^1, V^1)$$

are homotopic.

- (vi) $\mathbf{KK}(A, B)$ is defined to be the set of equivalence classes of $\mathrm{KK}(A, B)$ -cycles.

The following lemma is the real version of Lemma 2.3 of [15].

Lemma 2.4. $\mathbf{KK}(A, B)$ is an abelian group, for separable R^* -algebras A and B . As a functor it is contravariant in the first argument and covariant in the second argument.

Proof. For the first statement, we show that a cycle (ϕ_+, ϕ_-, U) has inverse (ϕ_-, ϕ_+, U^*) . Indeed, the sum

$$\left(\begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} \phi_- & 0 \\ 0 & \phi_+ \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \right)$$

is homotopic to a degenerate cycle via the operator homotopy

$$W_t = \begin{pmatrix} \cos(t)U & -\sin(t) \\ \sin(t) & \cos(t)U^* \end{pmatrix}, \quad t \in \left[0, \frac{\pi}{2}\right].$$

The functoriality is established as in [15] in Sections 2.4 through 2.7. \square

The next proposition establishes the isomorphism between the two pictures of KK-theory. First we review some preliminaries regarding graded R^* -algebras and Hilbert modules.

If B is an R^* -algebra, then the *standard even grading* on $M_2(B)$ is obtained by setting $M_2(B)^{(0)}$ to be the set of diagonal matrices and $M_2(B)^{(1)}$ the set of matrices with zero diagonal. The *standard even grading* on $\mathcal{K}_{\mathbb{R}} \otimes B$ is obtained by choosing a $*$ -isomorphism $\mathcal{K}_{\mathbb{R}} \otimes B \cong M_2(\mathcal{K}_{\mathbb{R}} \otimes B)$. This in turn induces a canonical (modulo a unitary automorphism) grading on $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$.

Let \mathbb{H}_B be the Hilbert B -module consisting of all sequences $\{b_n\}_{n=1}^{\infty}$ in B such that $\sum_{n=1}^{\infty} b_n^* b_n$ converges. Giving B the trivial grading (that is $B^{(0)} = B$ and $B^{(1)} = \{0\}$), let $\hat{\mathbb{H}}_B = \mathbb{H}_B \oplus \mathbb{H}_B$ be the graded Hilbert B -module with $\hat{\mathbb{H}}_B^{(0)} = \mathbb{H}_B \oplus 0$ and $\hat{\mathbb{H}}_B^{(1)} = 0 \oplus \mathbb{H}_B$. Then the induced grading on $\mathcal{L}(\hat{\mathbb{H}}_B)$ is identical with the standard even grading of $M_2(\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B))$. Under the $*$ -isomorphism $M_2(\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)) \cong \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$, this grading coincides with the one described in the previous paragraph.

Theorem 2.5. *Let A and B be separable R^* -algebras. Then $\text{KK}(A, B)$ is isomorphic to $\mathbf{KK}(A, B)$.*

Proof. We give A and B the trivial grading and we give $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ the standard even grading described above.

For a $\text{KK}(A, B)$ -cycle $x = (\phi_+, \phi_-, U)$ we define

$$\alpha(x) = \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right).$$

It is readily verified that $\alpha(x)$ is a Kasparov (A, B) bimodule. Furthermore, for $x = (\phi_+, \phi_-, U)$ and $y = (\psi_+, \psi_-, V)$ it is easy to see that $\alpha(x + y)$ and $\alpha(x) + \alpha(y)$ are unitarily equivalent via a degree 0 unitary.

We must show that α induces a well-defined homomorphism

$$\bar{\alpha}: \mathbf{KK}(A, B) \rightarrow \text{KK}(A, B).$$

Note first that α sends degenerate elements to degenerate elements. Next, suppose (ϕ_+, ϕ_-, U) is a $\text{KK}(A, IB)$ -cycle implementing a homotopy between $x = (\phi_+^0, \phi_-^0, U_0)$ and $y = (\phi_+^1, \phi_-^1, U_1)$. That is, $\varepsilon_i(\phi_+) = \phi_+^i$, $\varepsilon_i(\phi_-) = \phi_-^i$, and $\varepsilon_i(U) = U_i$; where $\varepsilon_t: \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes IB) \rightarrow \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ is the map induced by the evaluation map ε_t at t . Consider the Kasparov (A, IB) -bimodule

$$z = \left(\hat{\mathbb{H}}_{IB}, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right).$$

Since $\hat{\mathbb{H}}_{B \otimes C[0,1]} \hat{\otimes}_{\varepsilon_i} B \cong \hat{\mathbb{H}}_B$, it follows that

$$(\varepsilon_0)_*(z) = \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \phi_+^0 & 0 \\ 0 & \phi_-^0 \end{pmatrix}, \begin{pmatrix} 0 & U_0^* \\ U_0 & 0 \end{pmatrix} \right)$$

and

$$(\varepsilon_1)_*(z) = \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \phi_+^1 & 0 \\ 0 & \phi_-^1 \end{pmatrix}, \begin{pmatrix} 0 & U_1^* \\ U_1 & 0 \end{pmatrix} \right).$$

Therefore $\alpha(x)$ and $\alpha(y)$ are homotopic. Hence $\bar{\alpha}$ is well-defined.

To show that $\bar{\alpha}$ is surjective, let $y = (E, \phi, T)$ be a Kasparov (A, B) -bimodule. By Proposition 2.3.5 of [32], we may assume that $E \cong \hat{\mathbb{H}}_B$ and that $T = T^*$. Thus, with respect to the graded isomorphism $\mathcal{L}(\hat{\mathbb{H}}_B) \cong M_2(\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B))$, we can write

$$\phi = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where ϕ_+ and ϕ_- are $*$ -homomorphisms from A to $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ and U is an element of $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$. Then $y = \bar{\alpha}(x)$ where $x = (\phi_+, \phi_-, U)$.

Finally, we show that $\overline{\alpha}$ is injective. Suppose that $\overline{\alpha}(x) = \overline{\alpha}(y)$ where $x = (\phi_+, \phi_-, U)$ and $y = (\psi_+, \psi_-, V)$. Then there is a Kasparov (A, B) -bimodule $z = (E, \phi, T)$ such that

$$(\varepsilon_0)_*(z) \cong \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right)$$

and

$$(\varepsilon_1)_*(z) \cong \left(\hat{\mathbb{H}}_B, \begin{pmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{pmatrix}, \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix} \right).$$

As above, we may assume that $E = \hat{\mathbb{H}}_{B \otimes C[0,1]}$ and that $T = T^*$. Then z has the form

$$z = \left(\hat{\mathbb{H}}_{B \otimes C[0,1]}, \begin{pmatrix} \theta_+ & 0 \\ 0 & \theta_- \end{pmatrix}, \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} \right)$$

where θ_+ and θ_- are $*$ -homomorphisms to $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes IB)$ and W is an element of $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes IB)$. Then (θ_+, θ_-, W) is a Kasparov (A, IB) -bimodule implementing a homotopy between x and y . \square

3. THE UNIVERSAL PROPERTY OF KK-THEORY

Let F be a functor from the category $\mathbf{C}^*\mathbb{R}\text{-Alg}$ of separable R^* -algebras to the category \mathbf{Ab} of abelian groups. We say that F is

- (i) *homotopy invariant* if $(\alpha_1)_* = (\alpha_2)_*$ whenever α_1 and α_2 are homotopic $*$ -homomorphisms on the level of R^* -algebras.
- (ii) *stable* if $(e_A)_*: F(A) \rightarrow F(\mathcal{K}_{\mathbb{R}} \otimes A)$ is an isomorphism for the inclusion $e_A: A \hookrightarrow \mathcal{K}_{\mathbb{R}} \otimes A$ defined via any rank one projection.
- (iii) *split exact* if any split exact sequence of separable R^* -algebras

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

induces a split exact sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0.$$

- (iv) *half exact* if any short exact sequence of separable R^* -algebras

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

induces an exact sequence

$$F(A) \rightarrow F(B) \rightarrow F(C).$$

In what follows we will see that if F is homotopy invariant and half exact, then it is split exact.

Proposition 3.1. *If F is a functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, then the functor F_s defined by $F_s(A) = F(\mathcal{K}_{\mathbb{R}} \otimes A)$ is homotopy invariant and stable.*

Proof. Just as in the complex case (Theorem 4.1.13 of [19]), the map $e_{\mathcal{K}_{\mathbb{R}}}: \mathcal{K}_{\mathbb{R}} \rightarrow \mathcal{K}_{\mathbb{R}} \otimes \mathcal{K}_{\mathbb{R}}$ is homotopic to an isomorphism. \square

The following theorem is the version for R^* -algebras of Theorem 3.7 of [15] and Theorem 22.3.1 of [1].

Theorem 3.2. *Let F be a functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, stable, and split exact. Then there is a unique natural pairing $\alpha: F(A) \otimes \text{KK}(A, B) \rightarrow F(B)$ such that $\alpha(x \otimes 1_A) = x$ for all $x \in F(A)$ and where $1_A \in \text{KK}(A, A)$ is the class represented by the identity $*$ -homomorphism.*

Furthermore, the pairing respects the intersection product on KK-theory in the sense that

$$\alpha(\alpha(x \otimes y) \otimes z) = \alpha(x \otimes (y \otimes_B z)): F(A) \otimes \text{KK}(A, B) \otimes \text{KK}(B, C) \rightarrow F(C).$$

Proof. Let $\Phi \in \text{KK}(A, B)$. Using Theorem 2.5 we represent Φ with a $\text{KK}(A, B)$ cycle and as in Lemma 3.6 of [15], we may assume that this cycle has the form $(\phi_+, \phi_-, 1)$. We use the same construction as in Definitions 3.3 and 3.4 in [15]. In that setting F is assumed to be a functor from separable C^* -algebras, but it goes through the same for functors from separable R^* -algebras to any abelian category. This construction produces a homomorphism $\Phi_*: F(A) \rightarrow F(B)$ and we then define $\alpha(x \otimes \Phi) = \Phi_*(x)$. The proof of Theorems 3.7 and 3.5 of [15] carry over in the real case to show that α is natural, is well-defined, satisfies $\alpha(x \otimes 1_A) = x$, and is unique.

That α respects the Kasparov product follows from the uniqueness statement. \square

We also note the contravariant version of the result above. If F is a contravariant functor, otherwise satisfying the above hypotheses, then there is a pairing $\alpha: \text{KK}(A, B) \otimes F(B) \rightarrow F(A)$ such that $\alpha(1_A \otimes x) = x$ for all $x \in F(B)$.

For any R^* -algebra A , we define $SA = \{f \in C([0, 1], A) \mid f(0) = f(1) = 0\}$, or equivalently up to $*$ -isomorphism, $SA = C_0(\mathbb{R}, A)$. We similarly define $S^{-1}A = \{f \in C_0(\mathbb{R}, A) \mid f(-x) = \overline{f(x)}\}$. By iteration, $S^n A$ is defined for all $n \in \mathbb{Z}$. Since $SS^{-1}\mathbb{R}$ is KK -equivalent to \mathbb{R} , the formula $S^n S^m A \cong S^{n+m} A$ holds up to KK -equivalence for all $n, m \in \mathbb{Z}$. Then for any functor F on $\mathbf{C}^*\mathbf{R}\text{-Alg}$ and any integer n , we define $F_n(A) = F(S^n(A))$.

Corollary 3.3. *Let F be a functor from $\mathbf{C}^*\mathbf{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, stable, and split exact. Then $F_*(A)$ has the structure of a graded module over the ring $K_*(\mathbb{R})$. In particular, $F(S^8 A) \cong F(A)$ and $F(S^{-1}SA) \cong F(A)$.*

Proof. For all separable A and σ -unital B , the pairing of Proposition 2.2 gives $\text{KK}_*(A, B)$ the structure of a module over $\text{KK}_*(\mathbb{R}, \mathbb{R})$. Taking $A = B$, we define a graded ring homomorphism β from $K_*(\mathbb{R}) \cong \text{KK}_*(\mathbb{R}, \mathbb{R})$ to $\text{KK}_*(A, A)$ by multiplication by $1_A \in \text{KK}(A, A)$.

Then for any $x \in F_m(A)$ and $y \in K_n(\mathbb{R})$ we define $x \cdot y = \alpha(x \otimes \beta(y)) \in F_{n+m}(A)$. The second statement follows from the KK -equivalence between \mathbb{R} and $S^8\mathbb{R}$, and that between \mathbb{R} and $S^{-1}S\mathbb{R}$ from Section 1.4 of [2]. \square

It also follows that the pairing Theorem 3.2 extends to a well-defined graded pairing

$$\alpha: F_*(A) \otimes \text{KK}_*(A, B) \rightarrow F_*(B).$$

Let \mathbf{KK} be the category whose objects are separable R^* -algebras and the set of morphisms from A to B is $\text{KK}(A, B)$. There is a canonical functor KK from $\mathbf{C}^*\mathbf{R}\text{-Alg}$ to \mathbf{KK} that takes an object A to itself and which takes a $*$ -homomorphism $f: A \rightarrow B$ to the corresponding element $[f] \in \text{KK}(A, B)$.

Corollary 3.4. *Let F be a functor from $\mathbf{C}^*\mathbf{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, stable, and split exact. Then there exists a unique functor $\hat{F}: \mathbf{KK} \rightarrow \mathbf{Ab}$ such that $\hat{F} \circ \text{KK} = F$.*

Proof. The functor \hat{F} takes an object A in \mathbf{KK} to $F(A)$ in \mathbf{Ab} and takes a morphism $y \in \text{KK}(A, B)$ to the homomorphism $F(A) \rightarrow F(B)$ defined by $x \mapsto \alpha(x \otimes y)$. The composition $\hat{F} \circ \text{KK} = F$ clearly holds on the level of objects. On the level of morphisms we must verify the formula $\alpha(x \otimes [f]) = f_*(x)$ for $f: A \rightarrow B$ and $x \in F(A)$. This formula follows by the naturality of the pairing α , the formula $\alpha(x \otimes 1_A) = x$, and the formula $f_*(1_A) = [f] \in \text{KK}(A, B)$ which is verified as in Section 2.8 of [15]. \square

Proposition 3.5. *Let F be a functor from $\mathbf{C}^*\mathbf{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant and half exact. Then for any short exact sequence*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

there is a natural boundary map $\partial: F(SC) \rightarrow F(A)$ that fits into a (half-infinite) long exact sequence

$$\cdots \rightarrow F(SB) \xrightarrow{g_*} F(SC) \xrightarrow{\partial} F(A) \xrightarrow{f_*} F(B) \xrightarrow{g_*} F(C) \rightarrow \cdots$$

Proof. Use the mapping cone construction as in Section 21.4 of [1]. \square

Corollary 3.6. *A functor F from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant and half exact is also split exact.*

Proof. The splitting implies that g_* is surjective. Thus in the sequence of Proposition 3.5, $\partial = 0$ and f_* is injective. \square

Proposition 3.7. *Let F be a functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, stable, and half exact. Then for any short exact sequence*

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

there is a natural long exact sequence (with 24 distinct terms)

$$\cdots \rightarrow F_{n+1}(C) \xrightarrow{\partial} F_n(A) \xrightarrow{f_*} F_n(B) \xrightarrow{g_*} F_n(C) \xrightarrow{\partial} F_{n-1}(A) \rightarrow \cdots$$

Proof. From Corollary 3.6 and Corollary 3.3, F is periodic; so Proposition 3.5 gives the long exact sequence. \square

We say that a homotopy invariant, stable, half-exact functor F from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to the category \mathbf{Ab} of abelian groups

- (v) satisfies the *dimension axiom* if there is an isomorphism $F_*(\mathbb{R}) \cong K_*(\mathbb{R})$ as graded modules over $K_*(\mathbb{R})$
- (vi) is *continuous* if for any direct sequence of R^* -algebras (A_n, ϕ_n) , the natural homomorphism

$$\lim_{n \rightarrow \infty} F_*(A_n) \rightarrow F_*(\lim_{n \rightarrow \infty} (A_n))$$

is an isomorphism.

Theorem 3.8. *Let F be a functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that is homotopy invariant, stable, half exact and satisfies the dimension axiom. Then there is a natural transformation $\beta: K_n(A) \rightarrow F_n(A)$. If F is also continuous, then β is an isomorphism for all R^* -algebras in the smallest class of separable R^* -algebras which contains \mathbb{R} and is closed under KK-equivalence, countable inductive limits, and the two-out-of-three rule for exact sequences.*

Proof. Let z be a generator of $F(\mathbb{R}) \cong \mathbb{Z}$ and for $x \in K_n(A) \cong \text{KK}(\mathbb{R}, S^n A)$ define a $K_*(\mathbb{R})$ -module homomorphism $\beta: K_*(A) \rightarrow F_*(A)$ by $\beta(x) = \alpha(z \otimes x)$. Taking $A = \mathbb{R}$, Theorem 3.2 yields that $\beta(1_0) = z$ where 1_0 is the unit of the ring $K_*(\mathbb{R}) = \text{KK}_*(\mathbb{R}, \mathbb{R})$. Therefore, β is an isomorphism for $A = \mathbb{R}$. Then bootstrapping arguments show that β is an isomorphism for all R^* -algebras in the class described. \square

From Section 2.1 of [3] we have distinguished elements

$$\begin{aligned} c &\in \text{KK}_0(\mathbb{R}, \mathbb{C}), & r &\in \text{KK}_0(\mathbb{C}, \mathbb{R}) \\ \varepsilon &\in \text{KK}_0(\mathbb{R}, T), & \zeta &\in \text{KK}_0(T, \mathbb{C}) \\ \psi_U &\in \text{KK}_0(\mathbb{C}, \mathbb{C}), & \psi_T &\in \text{KK}_0(T, T) \\ \gamma &\in \text{KK}_{-1}(\mathbb{C}, T), & \tau &\in \text{KK}_1(T, \mathbb{R}). \end{aligned}$$

For any homotopy invariant, stable, split exact functor F on $\mathbf{C}^*\mathbb{R}\text{-Alg}$, define the united F -theory of an R^* -algebra A to be

$$F^{CRT}(A) = \{F_*(A), F_*(\mathbb{C} \otimes A), F_*(T \otimes A)\}$$

together with the collection of natural homomorphisms

$$\begin{aligned}
c_n &: F_n(A) \rightarrow F_n(\mathbb{C} \otimes A) \\
r_n &: F_n(\mathbb{C} \otimes A) \rightarrow F_n(A) \\
\varepsilon_n &: F_n(A) \rightarrow F_n(T \otimes A) \\
\zeta_n &: F_n(T \otimes A) \rightarrow F_n(\mathbb{C} \otimes A) \\
(\psi_U)_n &: F_n(\mathbb{C} \otimes A) \rightarrow F_n(\mathbb{C} \otimes A) \\
(\psi_T)_n &: F_n(T \otimes A) \rightarrow F_n(T \otimes A) \\
\gamma_n &: F_n(\mathbb{C} \otimes A) \rightarrow F_{n-1}(\mathbb{C} \otimes A) \\
\tau_n &: F_n(T \otimes A) \rightarrow F_{n+1}(A)
\end{aligned}$$

induced by the elements $c, r, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau$ via the pairing of Theorem 3.2.

Proposition 3.9. *Let F be a homotopy invariant, stable, split exact functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} and let A be a separable R^* -algebra. Then $F^{CRT}(A)$ is a CRT-module. Moreover, if in addition F is half exact, then $F^{CRT}(A)$ is acyclic.*

Proof. To show that $F^{CRT}(A)$ is a CRT-module, we must show that the CRT-module relations

$$\begin{array}{lll}
rc = 2 & \psi_U \beta_U = -\beta_U \psi_U & \xi = r\beta_U^2 c \\
cr = 1 + \psi_U & \psi_T \beta_T = \beta_T \psi_T & \omega = \beta_T \gamma \zeta \\
r = \tau \gamma & \varepsilon \beta_O = \beta_T^2 \varepsilon & \beta_T \varepsilon \tau = \varepsilon \tau \beta_T + \eta_T \beta_T \\
c = \zeta \varepsilon & \zeta \beta_T = \beta_U^2 \zeta & \varepsilon r \zeta = 1 + \psi_T \\
(\psi_U)^2 = 1 & \gamma \beta_U^2 = \beta_T \gamma & \gamma c \tau = 1 - \psi_T \\
(\psi_T)^2 = 1 & \tau \beta_T^2 = \beta_O \tau & \tau = -\tau \psi_T \\
\psi_T \varepsilon = \varepsilon & \gamma = \gamma \psi_U & \tau \beta_T \varepsilon = 0 \\
\zeta \gamma = 0 & \eta_O = \tau \varepsilon & \varepsilon \xi = 2\beta_T \varepsilon \\
\zeta = \psi_U \zeta & \eta_T = \gamma \beta_U \zeta & \xi \tau = 2\tau \beta_T
\end{array}$$

hold among the operations $\{c_n, r_n, \varepsilon_n, \zeta_n, (\psi_U)_n, (\psi_T)_n, \gamma_n, \tau_n\}$ on $F^{CRT}(A)$. But in the proof of Proposition 2.4 of [3], it is shown that these relations hold at the level of KK-elements. Therefore, using the associativity of the pairing of Theorem 3.2, the same relations hold among the operations of $F^{CRT}(A)$.

Suppose now that F is also half-exact. To show that $K^{CRT}(A)$ is acyclic, we must show that the sequences

$$\cdots \rightarrow F_n(A) \xrightarrow{\eta_O} F_{n+1}(A) \xrightarrow{c} F_{n+1}(\mathbb{C} \otimes A) \xrightarrow{r\beta_U^{-1}} F_{n-1}(A) \rightarrow \cdots \quad (3.1)$$

$$\cdots \rightarrow F_n(A) \xrightarrow{\eta_O^2} F_{n+2}(A) \xrightarrow{\varepsilon} F_{n+2}(T \otimes A) \xrightarrow{\tau\beta_T^{-1}} F_{n-1}(A) \rightarrow \cdots \quad (3.2)$$

$$\cdots \rightarrow F_{n+1}(\mathbb{C} \otimes A) \xrightarrow{\gamma} F_n(T \otimes A) \xrightarrow{\zeta} F_n(\mathbb{C} \otimes A) \xrightarrow{1-\psi_U} F_n(\mathbb{C} \otimes A) \rightarrow \cdots \quad (3.3)$$

are exact. These can be derived from the short exact sequences

$$0 \rightarrow S^{-1}\mathbb{R} \otimes A \rightarrow \mathbb{R} \otimes A \rightarrow \mathbb{C} \otimes A \rightarrow 0$$

$$0 \rightarrow S^{-2}\mathbb{R} \otimes A \rightarrow \mathbb{R} \otimes A \rightarrow T \otimes A \rightarrow 0$$

$$0 \rightarrow S\mathbb{C} \otimes A \rightarrow T \otimes A \rightarrow \mathbb{C} \otimes A \rightarrow 0$$

from Sections 1.2 and 1.4 of [2].

Indeed, focusing on the first one for our argument, the short exact sequence

$$0 \rightarrow S^{-1}\mathbb{R} \otimes A \xrightarrow{f} \mathbb{R} \otimes A \xrightarrow{g} \mathbb{C} \otimes A \rightarrow 0$$

gives rise to the long exact sequence

$$\cdots \rightarrow F_n(A) \xrightarrow{[f]} F_{n+2}(A) \xrightarrow{[g]} F_{n+2}(\mathbb{C} \otimes A) \xrightarrow{\partial} F_{n-1}(A) \rightarrow \cdots$$

where the homomorphisms are given by the multiplication of the elements $[f] \in \text{KK}_2(\mathbb{R}, \mathbb{R})$, $[g] \in \text{KK}_0(\mathbb{R}, \mathbb{C})$, and $\partial \in \text{KK}_{-1}(\mathbb{C}, \mathbb{R})$. In the case of the functor $\text{KK}(B, -)$, it was shown in the proof of Proposition 2.4 of [3] that the resulting sequence has the form

$$\cdots \rightarrow \text{KK}_n(B, A) \xrightarrow{\eta_O} \text{KK}_{n+1}(B, A) \xrightarrow{c} \text{KK}_{n+1}(B, \mathbb{C} \otimes A) \xrightarrow{r\beta_U^{-1}} \cdots$$

for all separable R^* -algebras B . It then follows easily that the KK-element equalities $[f] = \eta_O$, $[g] = c$, $\partial = r\beta_U^{-1}$ hold. This proves that Sequence 3.1 is exact. Sequences 3.2 and 3.3 are shown to be exact the same way. \square

Theorem 3.10. *Let F and G be homotopy invariant, stable, half exact functors from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} with a natural transformation $\mu_A: F(A) \rightarrow G(A)$. If μ_A is an isomorphism for all C^* -algebras A in $\mathbf{C}^*\mathbb{R}\text{-Alg}$, then μ_A is an isomorphism for all R^* -algebras in $\mathbf{C}^*\mathbb{R}\text{-Alg}$.*

Proof. Let A be a separable R^* -algebra. The natural transformation μ_A induces a homomorphism $\mu_A^{CRT}: F^{CRT}(A) \rightarrow G^{CRT}(A)$ of acyclic CRT-modules which is, by hypothesis, an isomorphism on the complex part. Then the results in Section 2.3 of [6] imply that μ_A^{CRT} is an isomorphism. \square

Corollary 3.11. *Any homotopy invariant, stable, half exact functor from $\mathbf{C}^*\mathbb{R}\text{-Alg}$ to \mathbf{Ab} that vanishes on all C^* -algebras vanishes on all R^* -algebras.*

4. APPLICATION: ISOMORPHISM BETWEEN KK_{-1} AND Ext

In this section, we will use Theorem 3.10 to show that $\text{KK}_{-1}(A, B)$ is naturally isomorphic to $\text{Ext}(A, B)^{-1}$ for separable R^* -algebras A and B .

Let $\mathfrak{e}: 0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$ be an exact sequence of R^* -algebras. As in the complex case, the associated mapping cone is defined to be

$$C_\pi = \{(e, f) \in E \oplus C_0([0, 1], B) \mid f(1) = 0, f(0) = \pi(e)\}$$

and there are $*$ -homomorphisms $\lambda_\mathfrak{e}: SA \rightarrow C_\pi$, $\gamma_\mathfrak{e}: B \rightarrow C_\pi$, and $\kappa_\mathfrak{e}: C_\pi \rightarrow E$ defined by

$$\lambda_\mathfrak{e}(f) = (0, f), \quad \gamma_\mathfrak{e}(b) = (\iota b, 0), \quad \text{and} \quad \kappa_\mathfrak{e}(e, f) = e.$$

As in the complex case, there is an exact sequence $0 \rightarrow SA \xrightarrow{\lambda_\mathfrak{e}} C_\pi \xrightarrow{\kappa_\mathfrak{e}} E \rightarrow 0$. Furthermore, $\kappa_\mathfrak{e} \circ \gamma_\mathfrak{e} = \iota$ and $[\gamma_\mathfrak{e}]$ is an invertible element of $\text{KK}_0(B, C_\pi)$.

Lemma 4.1. *Let $\mathfrak{e}: 0 \rightarrow B \rightarrow E \xrightarrow{\pi} A \rightarrow 0$ be an exact sequence of R^* -algebras. If \mathfrak{e} is trivial, i.e., there exists a $*$ -homomorphism $\sigma: A \rightarrow E$ such that $\pi \circ \sigma = \text{id}_A$. Then $[\lambda_\mathfrak{e}] = 0$ in $\text{KK}_0(SA, C_\pi)$.*

Proof. Because of the section map σ , it follows that π_* is surjective and that the boundary map vanishes in the sequence

$$\cdots \rightarrow \text{KK}_n(SA, B) \rightarrow \text{KK}_n(SA, E) \xrightarrow{\pi_*} \text{KK}_n(SA, A) \xrightarrow{\delta} \text{KK}_{n-1}(SA, B) \rightarrow \cdots$$

By the comments above, the sequence $0 \rightarrow SA \xrightarrow{\lambda_\mathfrak{e}} C_\pi \xrightarrow{\kappa_\mathfrak{e}} E \rightarrow 0$ induces the same long exact sequence on KK-theory. Thus the homomorphism $\text{KK}_n(SA, SA) \xrightarrow{(\lambda_\mathfrak{e})_*} \text{KK}_n(SA, C_\pi)$ is zero. Since $[\lambda_\mathfrak{e}] = (\lambda_\mathfrak{e})_*[1_{SA}]$ (where $[1_{SA}] \in \text{KK}_0(SA, SA)$) it follows that $[\lambda_\mathfrak{e}] = 0$. \square

Lemma 4.2. *Let $\mathfrak{e}_i : 0 \rightarrow B \rightarrow E_i \xrightarrow{\pi_i} A \rightarrow 0$ be an exact sequence of R^* -algebra with B stable and B an essential ideal of E_i . Suppose there exists a unitary $u \in \mathcal{M}(B)$ such that*

$$\text{Ad}(\pi_B(u)) \circ \tau_{\mathfrak{e}_1} = \tau_{\mathfrak{e}_2}$$

where $\pi_B : \mathcal{M}(B) \rightarrow \mathcal{Q}(B) = \mathcal{M}(B)/B$ is the canonical projection. Then $[\lambda_{\mathfrak{e}_1}] \times [\gamma_{\mathfrak{e}_1}]^{-1} = [\lambda_{\mathfrak{e}_2}] \times [\gamma_{\mathfrak{e}_2}]^{-1} \in \text{KK}_0(SA, B)$.

Proof. As in the complex case, the conjugacy of the Busby invariants implies the corresponding extensions are equivalent. That is, we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \xrightarrow{\pi_1} & A \longrightarrow 0 \\ & & \downarrow \text{Ad}(u) & & \downarrow \text{Ad}(u) & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & E_2 & \xrightarrow{\pi_2} & A \longrightarrow 0 \end{array}$$

where we are identifying E_i as a sub-algebra of $\mathcal{M}(B)$. Since the mapping cone construction is functorial, there exists a $*$ -isomorphism $\alpha : C_{\pi_1} \rightarrow C_{\pi_2}$ such that the diagram

$$\begin{array}{ccccc} SA & \xrightarrow{\lambda_{\mathfrak{e}_1}} & C_{\pi_1} & \xleftarrow{\gamma_{\mathfrak{e}_1}} & B \\ \parallel & & \downarrow \alpha & & \downarrow \text{Ad}(u) \\ SA & \xrightarrow{\lambda_{\mathfrak{e}_2}} & C_{\pi_2} & \xleftarrow{\gamma_{\mathfrak{e}_2}} & B. \end{array}$$

Since $[\text{Ad}(u)] = [1_B] \in \text{KK}_0(B, B)$, the result follows. \square

As in the complex case, we define the sum of two extensions in terms of the Busby invariants. Let B be stable and let $\pi_B : \mathcal{M}(B) \rightarrow \mathcal{Q}(B)$ be the canonical projection. Then the sum of two extensions with Busby invariants $\tau_i : A \rightarrow \mathcal{Q}(B)$ is the unique extension (up to strong equivalence) with Busby invariant $\tau_1 \oplus \tau_2 : A \rightarrow \mathcal{Q}(B)$. The $*$ -homomorphism $\tau_1 \oplus \tau_2$ is defined by fixing a $*$ -isomorphism $M_2(B) \cong B$. More specifically, $\tau_1 \oplus \tau_2 = \bar{\psi} \circ \sigma$ where $\sigma : A \rightarrow \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq \mathcal{Q}(B \oplus B)$ is defined by $\sigma = (\tau_1, \tau_2)$ and $\bar{\psi}$ is the composition

$$\mathcal{Q}(B \oplus B) \rightarrow M_2(\mathcal{Q}(B)) \xrightarrow{\cong} \mathcal{Q}(B).$$

Lemma 4.3. *Let $\mathfrak{e}_i : 0 \rightarrow B \rightarrow E_i \xrightarrow{\pi_i} A \rightarrow 0$ be an exact sequence of R^* -algebras with B stable. Let $\mathfrak{e}_1 \oplus \mathfrak{e}_2 : 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ be the sum as defined in the previous paragraph. Then*

$$[\lambda_{\mathfrak{e}_1 \oplus \mathfrak{e}_2}] \times [\gamma_{\mathfrak{e}_1 \oplus \mathfrak{e}_2}]^{-1} = [\lambda_{\mathfrak{e}_1}] \times [\gamma_{\mathfrak{e}_1}]^{-1} + [\lambda_{\mathfrak{e}_2}] \times [\gamma_{\mathfrak{e}_2}]^{-1} \in \text{KK}_0(SA, B).$$

Proof. Let τ_i be the Busby invariant corresponding to \mathfrak{e}_i and $\tau_1 \oplus \tau_2$ the Busby invariant of the sum $\mathfrak{e}_1 \oplus \mathfrak{e}_2$, as above. Let σ and $\bar{\psi}$ be as above also. Notice that σ is the Busby invariant of an extension of the form

$$0 \rightarrow B \oplus B \rightarrow E_\sigma \rightarrow A \rightarrow 0.$$

Since $\tau_1 \oplus \tau_2 = \bar{\psi} \circ \sigma$, there exist morphisms ψ and ϕ forming the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \oplus B & \longrightarrow & E_\sigma & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & E_{\mathfrak{e}_1 \oplus \mathfrak{e}_2} & \longrightarrow & A \longrightarrow 0. \end{array}$$

Let $\rho_i : B \oplus B \rightarrow B$ be the projection onto the i -coordinate and let $\bar{\rho}_i : \mathcal{Q}(B \oplus B) \rightarrow \mathcal{Q}(B)$ be the induced map. Then $\bar{\rho}_i \circ \sigma = \tau_i$. Hence, there exists a $*$ -homomorphisms $\phi_i : E_\sigma \rightarrow E_i$ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \oplus B & \longrightarrow & E_\sigma & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \rho_i & & \downarrow \phi_i & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & E_i & \longrightarrow & A \longrightarrow 0 \end{array}$$

commutes for $i = 1, 2$.

By the above paragraphs, the following diagrams

$$\begin{array}{ccc} SA & \xrightarrow{\lambda_\epsilon} & C_\rho \xleftarrow{\gamma_\epsilon} B \oplus B \\ \parallel & & \downarrow \tilde{\phi}_i \quad \downarrow \rho_i \\ SA & \xrightarrow{\lambda_{\epsilon_i}} & C_{\pi_i} \xleftarrow{\gamma_{\epsilon_i}} B \end{array} \quad \begin{array}{ccc} SA & \xrightarrow{\lambda_\epsilon} & C_\rho \xleftarrow{\gamma_\epsilon} B \oplus B \\ \parallel & & \downarrow \tilde{\phi} \quad \downarrow \psi \\ SA & \xrightarrow{\lambda_{\epsilon_1 \oplus \epsilon_2}} & C_{\pi_3} \xleftarrow{\gamma_{\epsilon_1 \oplus \epsilon_2}} B \end{array}$$

are commutative. Thus,

$$\begin{aligned} [\lambda_{\epsilon_1 \oplus \epsilon_2}][\gamma_{\epsilon_1 \oplus \epsilon_2}]^{-1} &= [\lambda_\epsilon][\tilde{\phi}][\gamma_{\epsilon_1 \oplus \epsilon_2}]^{-1} \\ &= [\lambda_\epsilon][\gamma_\epsilon]^{-1}[\psi] \\ &= [\lambda_\epsilon][\gamma_\epsilon]^{-1}([\rho_1] + [\rho_2]) \\ &= [\lambda_\epsilon]([\tilde{\phi}_1][\gamma_{\epsilon_1}]^{-1} + [\tilde{\phi}_2][\gamma_{\epsilon_2}]^{-1}) \\ &= [\lambda_{\epsilon_1}][\gamma_{\epsilon_1}]^{-1} + [\lambda_{\epsilon_2}][\gamma_{\epsilon_2}]^{-1}. \end{aligned}$$

□

The goal for the rest of this section is to produce an isomorphism between $\mathrm{KK}_{-1}(A, B)$ and the group $\mathrm{Ext}(A, B)^{-1}$ of invertible extensions. For $r, s \in \mathbb{Z}_{\geq 0}$, let $\mathcal{C}\ell_{r,s}$ denote the Clifford algebra $\mathcal{C}\ell(\mathbb{R}^{r+s}, Q_{r,s})$ where $Q_{r,s}$ is the quadratic form

$$Q_{r,s} = -\sum_{i=1}^r x_i^2 + \sum_{j=r+1}^{r+s} x_j^2.$$

We will be using the fact that $\mathrm{KK}_*(A \hat{\otimes} \mathcal{C}\ell_{p,q}, B \hat{\otimes} \mathcal{C}\ell_{r,s})$ depends, up to natural isomorphism, only on $(p-q) - (r-s)$ (see Corollary 4.2.10 of [32]). In particular, for any trivially graded R^* -algebras A and B , we have $\mathrm{KK}_{-1}(A, B) \cong \mathrm{KK}_0(A, B \hat{\otimes} \mathcal{C}\ell_{0,1})$.

A Kasparov module $(A, B \hat{\otimes} \mathcal{C}\ell_{0,1})$ -module is a triple from (ϕ, E, F) such that $\phi : A \rightarrow \mathbb{B}(E \hat{\otimes} \mathcal{C}\ell_{0,1})$ is a graded $*$ -homomorphism, F is an operator in $\mathbb{B}(E \hat{\otimes} \mathcal{C}\ell_{0,1})$ of degree 1, and

$$F\phi(a) - \phi(a)F, (F^2 - 1)\phi(a), (F - F^*)\phi(a)$$

are elements of $\mathcal{K}_{\mathbb{R}} \otimes \mathfrak{B}$ for all $a \in A$. Using an argument similar to that in the complex case, we may assume that $F^* = F = F^{-1}$ and $E = \mathbb{H}_B$ (see Section 17.4 and 17.6 of [1]). Since $\mathbb{B}(\mathbb{H}_B \hat{\otimes} \mathcal{C}\ell_{0,1}) \cong \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B) \oplus \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ with grading

$$\begin{aligned} (\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B) \oplus \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B))^{(0)} &= \{(x, x) : x \in \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)\} \\ (\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B) \oplus \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B))^{(1)} &= \{(x, -x) : x \in \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)\} \end{aligned}$$

we have that $\phi = \psi \oplus \psi$ for some $*$ -homomorphism $\psi : A \rightarrow \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$ and $F = T \oplus (-T)$ for some self-adjoint unitary T in $\mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B)$.

Set $\pi : \mathcal{M}(\mathcal{K}_{\mathbb{R}} \otimes B) \rightarrow \mathcal{Q}(\mathcal{K}_{\mathbb{R}} \otimes B)$ be the canonical projection. Set $p = \frac{T+1}{2}$. Then p is a projection and $\pi(p)(\pi \circ \psi(a)) = (\pi \circ \psi(a))\pi(p)$ for all $a \in A$. Define $\tau_{(E,\phi,F)} : A \rightarrow \mathcal{Q}(\mathcal{K}_{\mathbb{R}} \otimes B)$ by

$$\tau_{(E,\phi,F)}(a) = \pi(p\psi(a)p) .$$

Then $\tau_{(E,\phi,F)}$ is a $*$ -homomorphism from A to $\mathcal{Q}(\mathcal{K}_{\mathbb{R}} \otimes B)$. Set $\mathfrak{e}_{(E,\psi,F)}$ be the exact sequence

$$\mathfrak{e}_{(E,\psi,F)} : 0 \rightarrow \mathcal{K}_{\mathbb{R}} \otimes B \rightarrow X(E, \psi, F) \rightarrow A \rightarrow 0$$

induced by $\tau_{(E,\phi,F)}$. This construction is well-defined and gives a natural transformation

$$\Theta_{A,B} : \mathrm{KK}_{-1}(A, B) \rightarrow \mathrm{Ext}(A, B)^{-1} .$$

By Lemma 4.2, we also have a well-defined natural transformation

$$\Lambda_{A,B} : \mathrm{Ext}(A, B)^{-1} \rightarrow \mathrm{KK}(SA, B)$$

given by $\Lambda_{A,B}([\mathfrak{e}]) = [\lambda_{\mathfrak{e}_{(E,\psi,F)}}][\gamma_{\mathfrak{e}_{(E,\psi,F)}}]^{-1}$.

Theorem 4.4. *If A and B are separable R^* -algebras with B stable, then $\Theta_{A,B}$ and $\Lambda_{A,B}$ are isomorphisms.*

Proof. Let $\mu_{A,B} = \Lambda_{A,B} \circ \Theta_{A,B}$. For separable C^* -algebras A and B there is a similarly defined homomorphism $\mu_{A,B}^{\mathbb{C}} : \mathrm{KK}_{-1}^{\mathbb{C}}(A, B) \rightarrow \mathrm{KK}^{\mathbb{C}}(SA, B)$. According to Proposition 19.5.7 of [1], this composition $\mu_{A,B}$ is exactly the usual isomorphism from $\mathrm{KK}_{-1}^{\mathbb{C}}(A, B)$ to $\mathrm{KK}^{\mathbb{C}}(SA, B)$.

Suppose now that B is a separable C^* -algebra. Then as in Lemma 4.3 of [3], there is a natural isomorphism ι from $\mathrm{KK}_{*}^{\mathbb{C}}(A_{\mathbb{C}}, B)$ to $\mathrm{KK}_{*}(A, B)$. Furthermore, the diagram below commutes,

$$\begin{array}{ccc} \mathrm{KK}_{-1}^{\mathbb{C}}(A_{\mathbb{C}}, B) & \xrightarrow{\mu_{A_{\mathbb{C}}, B}^{\mathbb{C}}} & \mathrm{KK}^{\mathbb{C}}(SA_{\mathbb{C}}, B) \\ \downarrow \iota & & \downarrow \iota \\ \mathrm{KK}_{-1}(A, B) & \xrightarrow{\mu_{A, B}} & \mathrm{KK}(SA, B) \end{array}$$

showing that $\mu_{A,B}$ is an isomorphism when B is a C^* -algebra. It then follows by Theorem 3.10 that $\mu_{A,B}$ is an isomorphism for all separable R^* -algebras A and B .

It remains now to prove that $\Theta_{A,B}$ is surjective. Let $\tau : A \rightarrow \mathcal{Q}(B)$ be a $*$ -homomorphism such that there exists a $*$ -homomorphism $\tau^{-1} : A \rightarrow \mathcal{Q}(B)$ with $\tau \oplus \tau^{-1}$ is a trivial extension. Then there exists a $*$ -homomorphism

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} : A \rightarrow M_2(\mathcal{M}(B)) ,$$

which is a lifting of $\tau \oplus \tau^{-1}$. Set $q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathcal{M}(B))$. Then q is a projection with $q\phi(a) - \phi(a)q \in M_2(B)$. Moreover, if $\pi_2 : M_2(\mathcal{M}(B)) \rightarrow M_2(\mathcal{Q}(B))$ is the natural projection, then $\pi_2(q\phi(a)q) = \tau(a) \oplus 0$.

Let $\pi : \mathcal{M}(B) \rightarrow \mathcal{Q}(B)$ be the natural projection. Using the $*$ -isomorphism $M_2(\mathcal{M}(B)) \cong \mathcal{M}(B)$, we have a $*$ -homomorphism $\psi : A \rightarrow \mathcal{M}(B)$ and a projection $p \in \mathcal{M}(B)$ such that $p\psi(a) - \psi(a)p \in B$ and $\pi(p\psi(a)p) = \tau(a)$. Set $T = 2p - 1$. Then

$$\Theta_{A,B}((\mathbb{H}_B \hat{\otimes} \mathcal{C}\ell_{0,1}, \psi \oplus \psi, T \oplus -T)) = [\tau] .$$

Hence, $\Theta_{A,B}$ is surjective, which completes the proof. \square

5. ASYMPTOTIC MORPHISMS AND E -THEORY

In this section, we will use Theorem 3.10 to show that $\text{KK}(A, B)$ is naturally isomorphic to $E(A, B)$ for separable R^* -algebras A and B , when A is nuclear.

The following definition of an asymptotic morphism for R^* -algebras is exactly analogous to that for C^* -algebras. This definition also appears in Section 8 of [5].

Definition 5.1. Let A and B be R^* -algebras. An *asymptotic morphism* from A to B is a family $\langle \phi_t \rangle$, $(t \in [1, \infty))$ of maps from A to B with the following properties:

- (1) for all $a \in A$, $t \mapsto \phi_t(a)$ is bounded and continuous; and
- (2) The set $\langle \phi_t \rangle$ is asymptotically $*$ -linear and multiplicative, i.e.
 - (a) $\lim_{t \rightarrow \infty} \|\phi_t(\lambda a + b) - (\lambda \phi_t(a) + \phi_t(b))\| = 0$;
 - (b) $\lim_{t \rightarrow \infty} \|\phi_t(a^*) - \phi_t(a)^*\| = 0$; and
 - (c) $\lim_{t \rightarrow \infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$
 for all $a, b \in A$ and $\lambda \in \mathbb{R}$.

Two asymptotic morphisms $\langle \phi_t \rangle, \langle \psi_t \rangle : A \rightarrow B$ are *equivalent* if

$$\lim_{t \rightarrow \infty} [\phi_t(a) - \psi_t(a)] = 0$$

for all $a \in A$.

Definition 5.2. Let A be an R^* -algebra. Set

$$\begin{aligned} \beta B &= \{f : [1, \infty) \rightarrow B : f \text{ is bounded and continuous}\} \\ \beta_0 B &= \left\{f \in \beta B : \lim_{t \rightarrow \infty} f(t) = 0\right\} \\ \alpha B &= \beta B / \beta_0 B \end{aligned}$$

Let $\langle \phi_t \rangle : A \rightarrow B$ be an asymptotic morphism. By Property (1), for each $a \in A$, the function $\hat{a} : t \mapsto \phi_t(a)$ is an element of βB . Let $\rho_B : \beta B \rightarrow \alpha B$ be the natural projection. By Property (2), the map

$$a \mapsto \rho_B(\hat{a})$$

is a $*$ -homomorphism from A to αB , which we denote by $\overline{\langle \phi_t \rangle}$. If $\langle \phi_t \rangle$ is equivalent to $\langle \psi_t \rangle$, then $\overline{\langle \phi_t \rangle} = \overline{\langle \psi_t \rangle}$. Let $\phi : A \rightarrow \alpha B$ be a $*$ -homomorphism and let $\psi : A \rightarrow \beta B$ be a lifting of ϕ . Set $\phi_t = \text{ev}_t \circ \psi$. Then $\langle \phi_t \rangle$ is an asymptotic morphism from A to B such that $\overline{\langle \phi_t \rangle} = \phi$.

From the above paragraph, we have the following proposition (compare to Remark 25.1.4 (a) of [1]).

Proposition 5.3. *There is a bijection from the set of $*$ -homomorphism from A to αB to the set of equivalence classes of asymptotic morphisms from A to B .*

Remark 5.4. *Every $*$ -homomorphism from A to B defines an asymptotic morphism as follows: Let $\phi : A \rightarrow B$ be a $*$ -homomorphism. The ϕ induces an asymptotic morphism $\langle \phi_t \rangle$ by $\phi_t(a) = \phi(a)$ for all $t \in [1, \infty)$ and $a \in A$. We will denote this asymptotic morphism by $\langle \phi \rangle$.*

Definition 5.5. Let $\langle \phi_t \rangle, \langle \psi_t \rangle : A \rightarrow B$ be asymptotic morphisms. $\langle \phi_t \rangle$ is *homotopic* to $\langle \psi_t \rangle$ if there exists an asymptotic morphism $\langle \Phi_t \rangle : A \rightarrow IB$ such that for each $t \in [1, \infty)$ and $a \in A$,

$$\text{ev}_0^{[0,1]}(\Phi_t(a)) = \phi_t(a) \quad \text{and} \quad \text{ev}_1^{[0,1]}(\Phi_t(a)) = \psi_t(a).$$

Set $[[A, B]]$ to be the set of homotopy classes of asymptotic morphisms from A to B . If $\langle \phi_t \rangle$ is an asymptotic morphism, then $[\langle \phi_t \rangle]$ will denote the class in $[[A, B]]$ represented by $\langle \phi_t \rangle$. The following proposition is analogous to Remark 2.5.1.2(g) of [1].

Proposition 5.6. *Let $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ be equivalent asymptotic morphisms from A to B . Then $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ are homotopic.*

Proof. Let $a \in A$ and let $t \in [1, \infty)$. Set $\Phi_t(a)(s) = s\psi_t(a) + (1-s)\phi_t(a)$ for all $s \in [0, 1]$. Since $\Phi_t(a)$ is the straight line homotopy from $\phi_t(a)$ and $\psi_t(a)$, we have that $\Phi_t(a) \in IB$. Since

$$\|\Phi_t(a) - \Phi_{t'}(a)\| \leq \|\phi_t(a) - \phi_{t'}(a)\| + \|\psi_t(a) - \psi_{t'}(a)\| \quad (5.1)$$

for all $a \in A$ and for all $t, t' \in [1, \infty)$, we have that

$$t \mapsto \Phi_t(a)$$

is a continuous function for all $a \in A$.

Define $\Psi_t(a)(s) = \psi_t(a)$ for all $a \in A$, $t \in [1, \infty)$, and $s \in [0, 1]$. It is clear that $\langle \Psi_t \rangle$ is an asymptotic morphism from A to IB . Note that

$$\|\Phi_t(a) - \Psi_t(a)\| \leq \|\phi_t(a) - \psi_t(a)\|$$

for all $a \in A$ and $t \in [1, \infty)$. Thus, $\lim_{t \rightarrow [1, \infty)} (\Phi_t(a) - \Psi_t(a)) = 0$ which implies that $\langle \Phi_t \rangle$ is an asymptotic morphism from A and IB . By construction, $\text{ev}_0^{[0,1]} \circ \Phi_t = \phi_t$ and $\text{ev}_1^{[0,1]} \circ \Phi_t = \psi_t$ for all $t \in [1, \infty)$. \square

Let $s \in [0, 1]$. Then $\text{ev}_s^{[0,1]} : f \mapsto f(s)$ is a $*$ -homomorphism from IB to B . Define $\tilde{\text{ev}}_s^{[0,1]} : \beta IB \rightarrow \beta B$ by $\tilde{\text{ev}}_s^{[0,1]}(f)(t) = f(t)(s)$ for all $f \in \beta IB$. Let $f \in \beta IB$. Then the function

$$t \mapsto f(t)(s)$$

is continuous and bounded by $\|f\|$. Hence, $\tilde{\text{ev}}_s^{[0,1]}(f) \in \beta B$. Thus, $\tilde{\text{ev}}_s^{[0,1]} : \beta IB \rightarrow \beta B$ is a $*$ -homomorphism for all $s \in [0, 1]$. Note that $\tilde{\text{ev}}_s^{[0,1]}(\beta_0 IB) = \beta_0 B$. Thus, there exists a $*$ -homomorphism $\overline{\text{ev}}_s^{[0,1]} : \alpha IB \rightarrow \alpha B$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \beta_0 IB & \longrightarrow & \beta IB & \xrightarrow{\rho_{IB}} & \alpha IB \longrightarrow 0 \\ & & \downarrow \tilde{\text{ev}}_s^{[0,1]} & & \downarrow \tilde{\text{ev}}_s^{[0,1]} & & \downarrow \overline{\text{ev}}_s^{[0,1]} \\ 0 & \longrightarrow & \beta_0 B & \longrightarrow & \beta B & \xrightarrow{\rho_B} & \alpha B \longrightarrow 0 \end{array}$$

is commutative.

Proposition 5.7. *Let A and B be R^* -algebras and let $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ be asymptotic morphisms. Suppose there exists a $*$ -homomorphism $\Phi : A \rightarrow I\alpha B$ such that $\overline{\text{ev}}_0^{[0,1]} \circ \Phi = \overline{\langle \phi_t \rangle}$ and $\overline{\text{ev}}_1^{[0,1]} \circ \Phi = \overline{\langle \psi_t \rangle}$. Then $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ are homotopic.*

Proof. Note that there exists an asymptotic morphism $\langle \Phi_t \rangle : A \rightarrow IB$ such that $\overline{\langle \Phi_t \rangle} = \Phi$. Then $\langle \text{ev}_0^{[0,1]} \circ \Phi_t \rangle$ and $\langle \text{ev}_1^{[0,1]} \circ \Phi_t \rangle$ are asymptotic morphism from A to B that are homotopic. Since $\overline{\text{ev}}_0^{[0,1]} \circ \Phi = \overline{\langle \phi_t \rangle}$, we have that $\langle \phi_t \rangle$ and $\langle \text{ev}_0^{[0,1]} \circ \Phi_t \rangle$ are equivalent. Similarly, $\langle \psi_t \rangle$ and $\langle \text{ev}_1^{[0,1]} \circ \Phi_t \rangle$ are equivalent. By Proposition 5.6, $\langle \phi_t \rangle$ and $\langle \text{ev}_0^{[0,1]} \circ \Phi_t \rangle$ are homotopic and $\langle \psi_t \rangle$ and $\langle \text{ev}_1^{[0,1]} \circ \Phi_t \rangle$ are homotopic. Hence, $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ are homotopic. \square

Lemma 5.8. *Let I be an ideal of a separable R^* -algebra A . Then there exists a continuous approximate identity $\{u_t\}_{t \in [1, \infty)}$ of I which is quasi-central in A . In other words, for all t , u_t is positive and norm-bounded by 1; and for all $a \in A$, $\lim_{t \rightarrow \infty} \|u_t a - a u_t\| = 0$.*

Proof. Note that the complexification $I_{\mathbb{C}}$ of I is an ideal of the complexification $A_{\mathbb{C}}$ of A and $A_{\mathbb{C}}$ is separable since A is separable. Let τ be a conjugate-linear involution of $A_{\mathbb{C}}$ such that A can be identified with $\{x \in A_{\mathbb{C}} : \tau(x) = x\}$. Then I can be identified as $\{x \in I_{\mathbb{C}} : \tau(x) = x\}$. Since $I_{\mathbb{C}}$ is an ideal of a separable C^* -algebra $A_{\mathbb{C}}$, there exists a continuous quasi-central approximate identity $\{w_t\}_{t \in [1, \infty)}$ of $I_{\mathbb{C}}$.

Set $u_t = \frac{1}{2}(w_t + \tau(w_t)) \in A$. Then it is easy to verify that $\{u_t\}_{t \in [1, \infty)}$ is again a continuous quasi-central approximate identity of $I_{\mathbb{C}}$ (hence of I) and is again quasi-central in $A_{\mathbb{C}}$ (hence in A). \square

Lemma 5.9. (See Lemma 25.5.2 of [1]). Let A be an R^* -algebra. Let $\{u_t\}_{t \in [1, \infty)}$ be given such that u_t is positive and $\|u_t\| \leq 1$ for all t . Let $x \in A$ and let $f \in S\mathbb{R}$.

- (a) If $\lim_{t \rightarrow \infty} (xu_t - u_tx) = 0$, then $\lim_{t \rightarrow \infty} (xf(u_t) - f(u_t)x) = 0$.
- (b) If $\lim_{t \rightarrow \infty} u_tx = x$, then $\lim_{t \rightarrow \infty} f(u_t)x = 0$.

Proof. Note that,

$$u_t^k x - xu_t^k = \sum_{n=0}^{k-1} (u_t^{k-n} x u_t^n - u_t^{k-(n+1)} x u_t^{n+1}).$$

Therefore,

$$\|u_t^k x - xu_t^k\| \leq k \|u_t x - xu_t\|$$

which implies that $\lim_{t \rightarrow \infty} (u_t^k x - xu_t^k) = 0$, whenever $\lim_{t \rightarrow \infty} (xu_t - u_tx) = 0$.

Also,

$$u_t^k x - x = \sum_{n=0}^{k-1} (u_t^{k-n} x - u_t^{k-n-1} x).$$

Therefore,

$$\|u_t^k x - x\| \leq k \|u_t x - x\|$$

which implies that $\lim_{t \rightarrow [1, \infty)} (u_t^k x - x) = 0$ whenever $\lim_{t \rightarrow \infty} u_t x = x$.

The lemma now follows from the Stone-Weierstrass Theorem and the above observations. \square

Proposition 5.10. (See Proposition 25.5.1 in [1].) Let $\mathfrak{e} : 0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$ be an exact sequence of R^* -algebras. Suppose B has a continuous approximate identity $\{u_t\}_{t \in [1, \infty)}$ which is quasi-central in E . Let σ be a bounded continuous cross-section of π . Then there exists an asymptotic morphism $\langle \phi_t^{\mathfrak{e}} \rangle : SA \rightarrow B$ such that

$$\lim_{t \rightarrow \infty} (\phi_t^{\mathfrak{e}}(f \otimes a) - f(u_t)\sigma(a)) = 0,$$

for all $f \in S\mathbb{R}$ and $a \in A$, where we are identifying SA with $S\mathbb{R} \otimes A$. Moreover, if τ is another cross section of π , then the associated asymptotic morphisms are equivalent.

We note that the existence of the cross-section σ is given by the Bartle-Graves selection theorem.

Proof. Define $\theta_t : S\mathbb{R} \times A \rightarrow B$ by $\theta_t(f, x) = f(u_t)\sigma(x)$. Since $\{u_t\}_{t \in [1, \infty)}$ is continuous, for each $(f, x) \in S\mathbb{R} \times A$, the map $t \mapsto \theta_t(f, x)$ is continuous and bounded. Note that if τ is another cross section of π , then $\sigma(x) - \tau(x) \in B$ for all $x \in A$. Then for each $(f, x) \in S\mathbb{R} \times A$, by Lemma 5.9,

$$\lim_{t \rightarrow \infty} \|f(u_t)\sigma(x) - f(u_t)\tau(x)\| = 0.$$

Therefore, θ_t does not depend on the choice of cross section (up to equivalence of asymptotic morphisms). Let $x \in A$, then the function $f \mapsto \theta_t(f, x)$ is linear for each $t \in [1, \infty)$. Let $f \in S\mathbb{R}$. Since

$$\sigma(x + y) - \sigma(x) - \sigma(y), \sigma(x^*) - \sigma(x)^*, \text{ and } \sigma(\lambda x) - \lambda\sigma(x)$$

are elements of B for all $x, y \in A$ and $\lambda \in \mathbb{R}$, by Lemma 5.9

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\theta_t(f, x + y) - \theta_t(f, x) - \theta_t(f, y)\| &= 0 \\ \lim_{t \rightarrow \infty} \|\theta_t(f, x^*) - \theta_t(f, x)^*\| &= 0 \\ \lim_{t \rightarrow \infty} \|\theta_t(f, \lambda x) - \lambda\theta_t(f, x)\| &= 0 \end{aligned}$$

for all $x, y \in A$ and $\lambda \in \mathbb{R}$. Since $\sigma(xy) - \sigma(x)\sigma(y) \in B$ for all $x, y \in A$ and $(fg)(u_t) = f(u_t)g(u_t)$ for all $f, g \in S\mathbb{R}$ and $t \in [1, \infty)$ and since

$$\begin{aligned} &\|(fg)(u_t)\sigma(xy) - f(u_t)\sigma(x)g(u_t)\sigma(y)\| \\ &\leq \|(fg)(u_t)\sigma(xy) - (fg)(u_t)\sigma(x)\sigma(y)\| + \|f\| \|\sigma(y)\| \|g(u_t)\sigma(x) - \sigma(x)g(u_t)\| \end{aligned}$$

by Lemma 5.9,

$$\lim_{t \rightarrow \infty} \|\theta_t(fg, xy) - \theta_t(f, x)\theta_t(g, y)\| = 0$$

for all $f, g \in S\mathbb{R}$ and $x, y \in A$.

Since $\|f(u_t)\sigma(x)\| \leq \|f\| \|\sigma(x)\|$ for all $f \in S\mathbb{R}$ and $x \in A$ and since θ_t is independent of the choice of σ , we may choose a cross section of π such that $\|\sigma(x)\| = \|x\|$ for all $x \in A$. Therefore, θ_t defines a $*$ -homomorphism from $\psi : S\mathbb{R} \otimes A \rightarrow \alpha B$. Let $\phi : S\mathbb{R} \otimes A \rightarrow \beta B$ be a lifting of ψ . Set $\phi_t^\epsilon = \text{ev}_t \circ \phi : SA \rightarrow B$.

By construction,

$$\lim_{t \rightarrow \infty} (\phi_t^\epsilon(f \otimes a) - f(u_t)\sigma(a)) = 0,$$

for all $f \in S\mathbb{R}$ and $a \in A$. Also, by the above observation, τ is another cross section of π , then the induced $*$ -homomorphism from $S\mathbb{R} \otimes A$ to αB is ψ . Thus, the induced asymptotic morphism is equivalent to $\langle \phi_t^\epsilon \rangle$. \square

Proposition 5.11. *Let $\epsilon : 0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$ be an exact sequence of separable R^* -algebras. Then the class in $[[SA, B]]$ given by the asymptotic morphism $\langle \phi_t^\epsilon \rangle$ from Proposition 5.10 is independent of the choice of approximate identity that is quasi-central in E .*

Proof. Let $\{u_t\}_{t \in [1, \infty)}$ and $\{v_t\}_{t \in [1, \infty)}$ be continuous approximate identities for B that are quasi-central in E . Then $w_t(s) = su_t + (1 - s)v_t$ for all $s \in [0, 1]$ is an approximate identity for IB and is quasi-central in IE . Set $D = \{f \in IE : \pi(f(t)) = \pi(f(s)) \text{ for all } t, s \in [0, 1]\}$. Then D is a sub- R^* -algebra of IE , IB is an ideal of D , and $D/IB \cong A$. By Proposition 5.10, there exists an asymptotic morphism $\langle \Phi_t \rangle : SA \rightarrow IB$, built using the approximate identity $\{w_t\}$ and the cross-section $\sigma : A \rightarrow SB$. By construction, $\overline{\text{ev}}_0^{[0, 1]} \circ \overline{\langle \Phi_t \rangle}$ is equal to the $*$ -homomorphism from SA to B constructed from $\{u_t\}_{t \in [1, \infty)}$ and $\overline{\text{ev}}_0^{[0, 1]} \circ \overline{\langle \Phi_t \rangle}$ is equal to the $*$ -homomorphism from SA to B constructed from $\{v_t\}_{t \in [1, \infty)}$. By Proposition 5.7, we have that the asymptotic morphism constructed from $\{u_t\}_{t \in [1, \infty)}$ and $\{v_t\}_{t \in [1, \infty)}$ are homotopic. \square

As in the complex case there is a natural isomorphism between $[[A, \mathcal{K}_{\mathbb{R}} \otimes B]]$ and $[[\mathcal{K}_{\mathbb{R}} \otimes A, \mathcal{K}_{\mathbb{R}} \otimes B]]$ (see Section 25.4 of [1]). Also as in the complex case, the set $[[A, \mathcal{K}_{\mathbb{R}} \otimes SB]]$ has the structure of an abelian group. The group operation is defined equivalently either in terms of a chosen $*$ -isomorphism $M_2(\mathcal{K}_{\mathbb{R}}) \cong \mathcal{K}_{\mathbb{R}}$ or in terms of concatenation of loops in $\mathcal{K}_{\mathbb{R}} \otimes B$ based at 0.

Definition 5.12. For R^* -algebras A and B , we define $E(A, B) = [[SA, \mathcal{K}_{\mathbb{R}} \otimes SB]]$.

Given an asymptotic morphism $\langle \phi_t \rangle : A \rightarrow B$ and an element $\gamma \in SA$, the composition $\phi_t \circ \gamma$ is an element of SB for all $t \in [1, \infty)$. We let $S\phi_t$ denote the resulting map $SA \rightarrow SB$ and using the compactness of the circle, it is easy to show that $\langle S\phi_t \rangle$ is an asymptotic morphism. Furthermore it is easy to show if two asymptotic morphisms $\langle \phi_t \rangle$ and $\langle \psi_t \rangle$ are homotopic, then $\langle S\phi_t \rangle$ and $\langle S\psi_t \rangle$ are also homotopic. Thus, every asymptotic morphism from A to B induces a unique element of $E(A, B)$. It also follows that there is a well-defined map $\Sigma : E(A, B) \rightarrow E(SA, SB)$, which can easily be shown to be a group homomorphism.

In fact, the map Σ is a special case of the tensor product construction for asymptotic morphisms described in Lemma II.B.β.5 of [7], which carries over to the case of R^* -algebras. Given two asymptotic morphisms $\langle \phi_t \rangle : A \rightarrow C$ and $\langle \psi_t \rangle : B \rightarrow D$, there is a tensor product asymptotic morphism $\langle (\phi \otimes \psi)_t \rangle : A \otimes_{\max} B \rightarrow C \otimes_{\max} D$ which satisfies $\lim_{t \rightarrow \infty} ((\phi \otimes \psi)_t(a \otimes b) - \phi_t(a) \otimes \psi_t(b)) = 0$ for all $a \in A$ and $b \in B$.

We also will need to make use of the associative product structure on E -theory described in Proposition II.B.β.4 of [7], which also carries over to the case of R^* -algebras. Given two asymptotic morphisms $\langle \phi_t \rangle : A \rightarrow B$ and $\langle \psi_t \rangle : B \rightarrow C$, there is a composition asymptotic morphism $\langle \psi \circ \phi \rangle_t : A \rightarrow C$, defined uniquely up to homotopy. In the special case that ψ or ϕ is an actual $*$ -homomorphism, then this product is a literal composition. In the general case, a reparametrization is necessary to construct an asymptotic morphism from the composition (see Lemma II.B.β.3). The resulting product induces a natural homomorphism $E(A, B) \otimes E(B, C) \rightarrow E(A, C)$.

Theorem 5.13. *$E(A, B)$ is a bivariate functor from separable R^* -algebras to abelian groups. In both arguments, it is homotopy invariant, stable, half exact, and has a degree 8 periodicity isomorphism.*

Proof. The homotopy invariance is immediate and the stability follows from Proposition 3.1. By Propositions 5.10 and 5.11, any extension

$$\mathfrak{e} : 0 \rightarrow J \rightarrow A \xrightarrow{q} B \rightarrow 0$$

gives rise to a well-defined asymptotic morphism $\langle \phi_t^{\mathfrak{e}} \rangle$ from SB to J . Then the proofs leading up to and including Corollary 25.5.7 of [1] carry over to the real case to show that the functor $E(A, \cdot)$ is a split exact functor for fixed separable A . Then by Theorem 3.2 (of the present paper) there is a bilinear pairing $E(A, B) \otimes KK(B, C) \rightarrow E(A, C)$. Since this map is associative, multiplication by the Bott element in $KK(\mathbb{R}, S^8\mathbb{R})$ induces a periodicity isomorphism in the second argument of $E(\cdot, \cdot)$. Similarly, $E(\cdot, B)$ is also split exact, so by the comments following Theorem 3.2 there is a natural pairing, $KK(A, B) \otimes E(B, C) \rightarrow E(A, C)$ proving periodicity in the first argument.

We postpone the proof of half-exactness until after a series of lemmas. \square

For any elements $x \in E(A, B)$ and $z \in KK(B, C)$, the pairing described in the proof above gives an element in $E(A, C)$ which we will denote by $x \otimes_{\alpha} z$. Taking $1_A \in E(A, A)$ we obtain a homomorphism $\epsilon : KK(A, B) \rightarrow E(A, B)$. For $x \in E(A, B)$ and $y \in E(B, C)$, we let $x \otimes_E y$ denote the product in $E(A, C)$. Similarly for $z \in KK(A, B)$ and $w \in KK(B, C)$, we let $z \otimes_{KK} w$ denote the product in $KK(A, C)$.

Lemma 5.14. *For $x \in KK(A, B)$ and $y \in KK(B, C)$ we have $\epsilon(x \otimes_{KK} y) = \epsilon(x) \otimes_E \epsilon(y)$.*

Proof. First we show that $\epsilon(x \otimes_{KK} y) = \epsilon(x) \otimes_{\alpha} y$. Indeed, using Theorem 3.2 we have

$$\epsilon(x \otimes_{KK} y) = 1_A \otimes_{\alpha} (x \otimes_{KK} y) = (1_A \otimes_{\alpha} x) \otimes_{\alpha} y = \epsilon(x) \otimes_{\alpha} y.$$

Secondly, we prove that $z \otimes_{\alpha} y = z \otimes_E \epsilon(y)$ for $z \in E(A, B)$. Indeed, this clearly holds for $y = 1_B \in KK(B, B)$ so it holds in general by the uniqueness statement of Theorem 3.2.

From these facts, we complete the proof with the calculation

$$\epsilon(x \otimes_{KK} y) = \epsilon(x) \otimes_{\alpha} y = \epsilon(x) \otimes_E \epsilon(y).$$

\square

Lemma 5.15. *For any R^* -algebras, $\Sigma: E(A, B) \rightarrow E(SA, SB)$ is an isomorphism.*

Proof. Let $\alpha \in E(\mathbb{R}, S^8\mathbb{R})$ and $\beta \in E(S^8\mathbb{R}, \mathbb{R})$ be Bott elements arising from the corresponding Bott elements in KK-theory via ϵ . Then it follows from Lemma 5.14 that $\alpha \otimes_E \beta = 1_{\mathbb{R}}$ and $\beta \otimes \alpha = 1_{S^8\mathbb{R}}$. It follows that the map $z \mapsto (\alpha \otimes 1_A) \otimes_E z \otimes (\beta \otimes 1_B)$ is an isomorphism from $E(S^8A, S^8B)$ to $E(A, B)$. We call this isomorphism Θ .

Now consider the homomorphism $\Sigma^8: E(A, B) \rightarrow E(S^8A, S^8B)$. From the comments following Definition 5.12, this homomorphism can be expressed as $z \mapsto 1_{S^8\mathbb{R}} \otimes z$. Then for $z \in E(A, B)$ the composition is given by

$$\begin{aligned} (\Theta \circ \Sigma^8)(z) &= (\alpha \otimes 1_A) \otimes_E (1_{S^8\mathbb{R}} \otimes z) \otimes_E (\beta \otimes 1_B) \\ &= (\alpha \otimes_E \beta) \otimes z \\ &= 1_{\mathbb{R}} \otimes z \\ &= z. \end{aligned}$$

It follows that $\Sigma: E(A, B) \rightarrow E(SA, SB)$ is injective.

Since $\Theta \circ \Sigma^8 = \text{id}_{E(A, B)}$ and since Θ is already known to be an isomorphism, it also follows that $\Sigma^8 \circ \Theta = \text{id}_{E(S^8A, S^8B)}$. Hence $\Sigma: E(S^7A, S^7B) \rightarrow E(S^8A, S^8B)$ is surjective.

Replacing A and B with suspensions of appropriate degrees, we obtain that $\Sigma: E(S^8A, S^8B) \rightarrow E(S^9A, S^9B)$ is an isomorphism.

Finally, consider the diagram

$$\begin{array}{ccc} E(A, B) & \xrightarrow{\Sigma} & E(SA, SB) \\ \Theta \uparrow & & \uparrow \Theta \\ E(S^8A, S^8B) & \xrightarrow{\Sigma} & E(S^9A, S^9B) \end{array}$$

where the vertical maps and the lower map are all known to be isomorphisms. This diagram commutes modulo a homomorphism induced by the rearrangement of the order of the suspension factors of S^9A and S^9B . Since the rearrangement of the factors corresponds to an even permutation, it is homotopic to the identity and induces an identity homomorphism on E -theory. It follows that the $\Sigma: E(A, B) \rightarrow E(SA, SB)$ is an isomorphism. \square

Completion of proof of Theorem 5.13. Finally, to prove half-exactness let $0 \rightarrow J \xrightarrow{\iota} A \xrightarrow{q} B \rightarrow 0$ be an extension of separable R^* -algebras and D be an R^* -algebra. Suppose h is an asymptotic morphism from $SD \otimes \mathcal{K}_{\mathbb{R}}$ to $SA \otimes \mathcal{K}_{\mathbb{R}}$ such that $[q \circ h] = [0]$. Lemma 25.5.12 of [1] and its proof carry over to the real case, so there exists an asymptotic morphism k from $S^2D \otimes \mathcal{K}_{\mathbb{R}}$ to $S^2J \otimes \mathcal{K}_{\mathbb{R}}$ such that $[S\iota \circ k] = [Sh]$. In the commutative diagram

$$\begin{array}{ccccc} E(D, J) & \xrightarrow{\iota_*} & E(D, A) & \xrightarrow{q_*} & E(D, B) \\ \downarrow \Sigma & & \downarrow \Sigma & & \downarrow \Sigma \\ E(SD, SJ) & \xrightarrow{\iota_*} & E(SD, SA) & \xrightarrow{q_*} & E(SD, SB) \end{array}$$

the vertical maps are isomorphisms, so there exists an asymptotic morphism g from $SD \otimes \mathcal{K}_{\mathbb{R}}$ to $SJ \otimes \mathcal{K}_{\mathbb{R}}$ such that $[\iota \circ g] = [h]$ in $E(D, A)$.

Half-exactness in the first argument is proved in a similar way. \square

Theorem 5.16. *Let A be a separable, nuclear R^* -algebra and let B be a separable R^* -algebra. Then the homomorphism $\epsilon: \text{KK}(A, B) \rightarrow E(A, B)$ is an isomorphism.*

Proof. By Theorem 3.10, it suffices to show that $\mathrm{KK}(A, B) \rightarrow E(A, B)$ is an isomorphism when B is a C^* -algebra. In the diagram

$$\begin{array}{ccc} \mathrm{KK}^{\mathbb{C}}(A_{\mathbb{C}}, B) & \longrightarrow & E^{\mathbb{C}}(A_{\mathbb{C}}, B) \\ \downarrow & & \downarrow \\ \mathrm{KK}(A, B) & \longrightarrow & E(A, B) \end{array}$$

we use $\mathrm{KK}^{\mathbb{C}}(-, -)$ and $E^{\mathbb{C}}(-, -)$ to denote the versions of these functors on C^* -algebras (for example $E^{\mathbb{C}}(-, -)$ consists of homotopy classes of asymptotic morphisms that are asymptotically linear over \mathbb{C}). Since $A_{\mathbb{C}}$ is nuclear, the top horizontal homomorphism is an isomorphism by Theorem 25.6.3 of [1]. The left vertical homomorphism is an isomorphism by Lemma 4.3 of [3]. The right vertical homomorphism is defined by restriction—a complex asymptotic morphism defined on $SA_{\mathbb{C}} \otimes \mathcal{K}$ restricts to a real asymptotic morphism defined on $SA \otimes \mathcal{K}_{\mathbb{R}}$ —and it is an isomorphism since every real asymptotic morphism on $SA \otimes \mathcal{K}_{\mathbb{R}}$ can be extended uniquely to a complex asymptotic morphism on $SA_{\mathbb{C}} \otimes \mathcal{K}$. The square commutes by Theorem 3.7 of [15], since the two directions around the square give natural transformations $\mathrm{KK}^{\mathbb{C}}(A_{\mathbb{C}}, -) \rightarrow E(A, -)$ of functors defined on separable C^* -algebras, each of which sends $1_A \in \mathrm{KK}^{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$ to the asymptotic morphism represented by the inclusion of A into $A_{\mathbb{C}}$. Therefore, the bottom row is an isomorphism as desired. \square

Finally, we present below the real analog of Theorem 5.8 of [28], showing that E -theory is a special case of KK -theory. That we can obtain this result without reproducing the extensive technicalities of the proof in [28] shows the utility of Theorem 3.10.

Since $\mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}})$ is KK -trivial, it follows that $E(A, \mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}})) = 0$. Then we use the long exact sequence arising from

$$0 \rightarrow B \otimes \mathcal{K}_{\mathbb{R}} \rightarrow \mathcal{M}(B \otimes \mathcal{K}_{\mathbb{R}}) \rightarrow \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}}) \rightarrow 0$$

to get an isomorphism $E_0(A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) \cong E_{-1}(A, B \otimes \mathcal{K}_{\mathbb{R}})$. Combining this with stability and with Lemma 5.15, there is an isomorphism

$$\gamma: E(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) \rightarrow E(A, B)$$

for R^* -algebras A and B .

Theorem 5.17. *Let A and B be a separable R^* -algebras. Then there is an isomorphism*

$$\varepsilon': \mathrm{KK}(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) \rightarrow E(A, B).$$

Proof. Let $\varepsilon' = \gamma \circ \varepsilon$ where ε is the isomorphism of Theorem 5.16. We know that ε' is an isomorphism in the complex case by Theorem 5.8 of [28]. So it suffices by Theorem 3.10 to show that ε' fits in a commutative square in the same way that ε does in the proof of Theorem 5.16. The square in question can be factored into two squares as follows:

$$\begin{array}{ccccc} \mathrm{KK}^{\mathbb{C}}(S^{-1}A_{\mathbb{C}}, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) & \longrightarrow & E^{\mathbb{C}}(S^{-1}A_{\mathbb{C}}, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) & \longrightarrow & E^{\mathbb{C}}(A_{\mathbb{C}}, B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{KK}(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) & \longrightarrow & E(S^{-1}A, \mathcal{Q}(B \otimes \mathcal{K}_{\mathbb{R}})) & \longrightarrow & E(A, B) \end{array}$$

The first square is just a specialization of the square in Theorem 5.16. The second square commutes since the horizontal maps are homomorphisms that arise from stabilization, from long exact sequences, and from suspensions; all of which commute with the vertical restriction map. \square

6. APPLICATION: ASYMPTOTIC MORPHISMS ON SPHERES

The goal of this section is to prove the following theorem, using the results of the previous sections, the universal coefficient theorem for real C^* -algebras, and a united K -theory analysis.

Theorem 6.1. *There exists asymptotic morphisms $\langle \phi_t \rangle$ that induce non-trivial homomorphisms on K -theory of the following forms:*

- (1) $S^d \mathbb{R} \rightarrow \mathcal{K}_{\mathbb{R}}$ where $d > 0$ and $d \equiv 0, 4, 6, 7 \pmod{8}$,
- (2) $S^d \mathbb{R} \rightarrow \mathcal{K}_{\mathbb{R}} \otimes \mathbb{H}$ where $d > 0$ and $d \equiv 0, 2, 3, 4 \pmod{8}$

Furthermore, if $n \in \mathbb{N}$ does not satisfy the specified condition in each case, then there does not exist a corresponding asymptotic morphism that is non-trivial on K -theory.

Before we prove Theorem 6.1 we need to establish some relationships between asymptotic morphisms and mapping cones which are well-known for C^* -algebras. Note that if $\phi : B \rightarrow C$ is a $*$ -homomorphism and $\langle \psi_t \rangle : A \rightarrow B$ is an asymptotic morphism, then $\langle \phi \circ \psi_t \rangle : A \rightarrow C$ is an asymptotic morphism.

Proposition 6.2. *Let $\epsilon : 0 \rightarrow B \xrightarrow{\iota} E \xrightarrow{\pi} A \rightarrow 0$ be a short exact sequence of R^* -algebras. Then $[\langle \gamma_{\epsilon} \circ \phi_t^{\epsilon} \rangle] = [\langle \lambda_{\epsilon} \rangle]$ in $[[SA, C_{\pi}]]$, where $\langle \phi_t^{\epsilon} \rangle$ is the asymptotic morphism associated to ϵ .*

Proof. Let $\sigma : A \rightarrow E$ be a cross section of π such that $\|\sigma(x)\| = \|x\|$. Let $\{u_t\}_{t \in [1, \infty)}$ be an approximate identity of B that is quasi-central in E . For $f \in S\mathbb{R}$ and $x \in A$ and $s \in [0, 1]$,

$$\begin{aligned}\phi_t(f, x)(s) &= f(su_t + (1-s)1_{\tilde{E}})\sigma(x) \\ \psi_t(f, x)(s)(r) &= f(r+1-s)x, \quad r \in [0, 1]\end{aligned}$$

where $\tilde{E} = E$ if E unital and \tilde{E} is the unitization of E if E is not unital, and $f(r+1-s) = 0$ if $r+1-s \geq 1$. By the continuity of continuous functional calculus, we have that $\psi_t(f, x)$ is continuous and $\phi_t(f, x)$ is continuous. Since

$$\begin{aligned}\pi(\phi_t(f, x)(s)) &= f(1-s)\pi(\sigma(x)) = f(1-s)x = \psi_t(f, x)(s)(0) \\ \text{and} \\ \psi_t(f, x)(s)(1) &= 0,\end{aligned}$$

we have that $(\phi_t(f, x), \psi_t(f, x)) \in IC_{\pi}$.

For all $f \in S\mathbb{R}$ and $x \in A$, it is clear that $t \mapsto \psi_t(f, x)$ is continuous and bounded in ICA and that $t \mapsto \phi_t(f, x)$ is the same in IE . Hence, the map $t \mapsto (\phi_t(f, x), \psi_t(f, x)) \in IC_{\pi}$, is continuous and bounded on $[1, \infty)$.

Define $\Psi : S\mathbb{R} \otimes A \rightarrow \beta IC_{\pi}$ by $\Psi(f, x)(t) = (\phi_t(f, x), \psi_t(f, x))$. Let $b \in B$, $x \in E$, and $t \in [1, \infty)$. Then

$$\begin{aligned}\sup \{ \|((1-s)u_t + s1_{\tilde{E}})b - b\| : s \in [0, 1] \} &= \sup \{ \|(1-s)u_t b - (1-s)b\| : s \in [0, 1] \} \\ &= \|u_t b - b\|\end{aligned}$$

and

$$\begin{aligned}\sup \{ \|((1-s)u_t + s1_{\tilde{E}})x - x((1-s)u_t + s1_{\tilde{E}})\| : s \in [0, 1] \} \\ = \sup \{ \|(1-s)(u_t x - x u_t)\| : s \in [0, 1] \} \\ = \|u_t x - x u_t\|.\end{aligned}$$

Set $f_t \in I\tilde{E}$ by $f_t(s) = (1-s)u_t + s1_{\tilde{E}}$. Then by the above paragraph, $\{f_t\}_{t \in [1, \infty)}$ is a family of positive elements in $I\tilde{E}$ with $\|f_t\| \leq 1$ such that

$$\lim_{t \rightarrow \infty} f_t b = b$$

for all $b \in B$ and

$$\lim_{t \rightarrow \infty} (f_t x - x f_t) = 0$$

for all $x \in E$, where we identify b and x in $I\tilde{E}$ as constant functions. Therefore, for all $f, g \in S\mathbb{R}$, for all $x, y \in A$, and for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \lim_{t \rightarrow \infty} (\phi_t(f, x + y) - \phi_t(f, x) - \phi_t(f, y)) &= 0 \\ \lim_{t \rightarrow \infty} (\phi_t(f, x^*) - \phi_t(f, x)^*) &= 0 \\ \lim_{t \rightarrow \infty} (\phi_t(f, \lambda x) - \lambda \phi_t(f, x)) &= 0 \\ \lim_{t \rightarrow \infty} (\phi_t(fg, xy) - \phi_t(f, x)\phi_t(g, y)) &= 0, \end{aligned}$$

using Lemma 5.9 as in the proof of Proposition 5.10.

By the definition of ψ_t , for all $f, g \in S\mathbb{R}$, for all $x, y \in A$, and for all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \lim_{t \rightarrow \infty} (\psi_t(f, x + y) - \psi_t(f, x) - \psi_t(f, y)) &= 0 \\ \lim_{t \rightarrow \infty} (\psi_t(f, x^*) - \psi_t(f, x)^*) &= 0 \\ \lim_{t \rightarrow \infty} (\psi_t(f, \lambda x) - \lambda \psi_t(f, x)) &= 0 \\ \lim_{t \rightarrow \infty} (\psi_t(fg, xy) - \psi_t(f, x)\psi_t(g, y)) &= 0. \end{aligned}$$

It is clear from the definition of both ϕ_t and ψ_t that $\phi_t(\cdot, x)$ and $\psi_t(\cdot, x)$ are linear functions.

From the above observations, there exists a $*$ -homomorphism $\Phi : S\mathbb{R} \otimes A \rightarrow \alpha IC_\pi$ such that if $\tilde{\Phi}$ is a set-theoretical lifting of Φ and $\Phi_t = \text{ev}_t \circ \tilde{\Phi}$ is the associated asymptotic morphism, then for each $f \in S\mathbb{R}$ and $x \in A$,

$$\lim_{t \rightarrow \infty} (\Phi_t(f \otimes x) - \Psi(f, x)(t)).$$

Note that

$$\begin{aligned} \text{ev}_0^{[0,1]}(\Psi(f, x)(t)) &= (\phi_t(f, x)(0), \psi_t(f, x)(0)) \\ &= (f(u_t)\sigma(x), 0) \\ &= \gamma_\epsilon(f(u_t)\sigma(x)) \end{aligned}$$

and

$$\begin{aligned} \text{ev}_1^{[0,1]}(\Psi(f, x)(t)) &= (\phi_t(f, x)(1), \psi_t(f, x)(1)) \\ &= (0, f \otimes x) \\ &= \lambda_\epsilon(f \otimes x). \end{aligned}$$

Hence, for all $f \in S\mathbb{R}$ and $x \in A$,

$$\lim_{t \rightarrow \infty} \left\| \text{ev}_0^{[0,1]} \circ \Phi_t(f \otimes x) - \gamma_\epsilon \circ \phi_t^\epsilon(f \otimes x) \right\| = \lim_{t \rightarrow \infty} \left\| \text{ev}_0^{[0,1]} \circ (\Phi_t(f \otimes x) - \Psi(f, x)(t)) \right\| = 0$$

and

$$\lim_{t \rightarrow \infty} \left\| \text{ev}_1^{[0,1]} \circ \Phi_t(f \otimes x) - \lambda_\epsilon(f \otimes x) \right\| = \lim_{t \rightarrow \infty} \left\| \text{ev}_1^{[0,1]} \circ (\Phi_t(f \otimes x) - \Psi(f, x)(t)) \right\| = 0.$$

Thus, $\langle \text{ev}_0^{[0,1]} \circ \Phi_t \rangle$ is homotopic to $\langle \gamma_\epsilon \circ \phi_t^\epsilon \rangle$ and $\langle \text{ev}_1^{[0,1]} \circ \Phi_t \rangle$ is homotopic to $\langle \lambda_\epsilon \rangle$. Since the asymptotic morphism $\langle \Phi_t \rangle$ gives a homotopy between $\langle \text{ev}_0^{[0,1]} \circ \Phi_t \rangle$ and $\langle \text{ev}_1^{[0,1]} \circ \Phi_t \rangle$, we have that $\langle \gamma_\epsilon \circ \phi_t^\epsilon \rangle$ and $\langle \lambda_\epsilon \rangle$ are homotopic. Thus, $[\langle \gamma_\epsilon \circ \phi_t^\epsilon \rangle] = [\langle \lambda_\epsilon \rangle]$ in $[[SA, C_\pi]]$. \square

Let B be an R^* -algebra. Then we have the following exact sequence of R^* -algebras

$$0 \rightarrow \beta_0 B \rightarrow \beta B \xrightarrow{\rho_B} \alpha B \rightarrow 0.$$

The above exact sequence induces a long exact sequence in K -theory

$$\cdots \rightarrow K_*(\beta_0 B) \rightarrow K_*(\beta B) \xrightarrow{(\rho_B)_*} K_*(\alpha B) \rightarrow K_{*-1}(\beta_0 B) \rightarrow \cdots$$

for each $n \in \mathbb{Z}$. Since $\beta_0 B$ is contractible, $K_*(\beta_0 B) = 0$. Hence, $(\rho_B)_*$ is an isomorphism.

Definition 6.3. Let A and B be R^* -algebras and let $\langle \phi_t \rangle : A \rightarrow B$ be an asymptotic morphism. Let $\langle \phi_t \rangle_*$ denote the composition

$$(\text{ev}_1^{[1,\infty)})_* \circ (\rho_B)_*^{-1} \circ \overline{\langle \phi_t \rangle}_*$$

from $K_*(A)$ to $K_*(B)$.

The next three lemmas are well-known in the context of C^* -algebras. We are not able to find a reference for R^* -algebras so for the convenience of the reader we provided the proof here.

Lemma 6.4. *Let A and B be R^* -algebras and let $\langle \phi_t \rangle, \langle \psi_t \rangle : A \rightarrow B$ be asymptotic morphisms. If $\langle \phi_t \rangle$ is homotopic to $\langle \psi_t \rangle$, then $\langle \phi_t \rangle_* = \langle \psi_t \rangle_*$.*

Proof. Let $\langle \Phi_t \rangle : A \rightarrow C([0,1], B)$ be an asymptotic morphism such that $\text{ev}_0^{[0,1]} \circ \Phi_t = \phi_t$ and $\text{ev}_1^{[0,1]} \circ \Phi_t = \psi_t$ for all $t \in [1, \infty)$. Since

$$(\text{ev}_s^{[0,1]})_* \langle \Phi_t \rangle_* = (\text{ev}_s^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_* (\rho_{IB})_*^{-1} \overline{\langle \Phi_t \rangle}_*$$

for all $s \in [0,1]$ and since for each $f \in \beta IB$, the function

$$s \mapsto \text{ev}_s^{[0,1]} \circ \text{ev}_1^{[1,\infty)}(f)$$

is continuous,

$$(\text{ev}_0^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_* = (\text{ev}_1^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_*.$$

Therefore,

$$\begin{aligned} (\text{ev}_0^{[0,1]})_* \langle \Phi_t \rangle_* &= (\text{ev}_0^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_* (\rho_{IB})_*^{-1} \overline{\langle \Phi_t \rangle}_* \\ &= (\text{ev}_1^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_* (\rho_{IB})_*^{-1} \overline{\langle \Phi_t \rangle}_* \\ &= (\text{ev}_1^{[0,1]})_* \langle \Phi_t \rangle_* . \end{aligned}$$

Note that $\overline{\text{ev}_0^{[0,1]} \circ \langle \Phi_t \rangle} = \overline{\langle \phi_t \rangle}$, $\overline{\text{ev}_1^{[0,1]} \circ \langle \Phi_t \rangle} = \overline{\langle \psi_t \rangle}$, $\overline{\text{ev}_s^{[0,1]} \circ \rho_{IB}} = \rho_B \circ \tilde{\text{ev}}_s^{[0,1]}$, and $\text{ev}_1^{[1,\infty)} \circ \tilde{\text{ev}}_s^{[0,1]} = \text{ev}_s^{[0,1]} \circ \text{ev}_1^{[1,\infty)}$. Therefore,

$$\begin{aligned} (\text{ev}_0^{[0,1]})_* \langle \Phi_t \rangle_* &= (\text{ev}_0^{[0,1]})_* (\text{ev}_1^{[1,\infty)})_* (\rho_{IB})_*^{-1} \overline{\langle \Phi_t \rangle}_* \\ &= (\text{ev}_1^{[1,\infty)})_* (\tilde{\text{ev}}_0^{[0,1]})_* (\rho_{IB})_*^{-1} \overline{\langle \Phi_t \rangle}_* \\ &= (\text{ev}_1^{[1,\infty)})_* (\rho_B)_*^{-1} (\overline{\text{ev}}_0^{[0,1]})_* \overline{\langle \Phi_t \rangle}_* \\ &= (\text{ev}_1^{[1,\infty)})_* (\rho_B)_*^{-1} \overline{\langle \phi_t \rangle}_* \\ &= \langle \phi_t \rangle_* . \end{aligned}$$

A similar computation shows that

$$(\text{ev}_1^{[0,1]})_* \langle \Phi_t \rangle_* = \langle \psi_t \rangle_* ,$$

which implies that

$$\langle \phi_t \rangle_* = (\text{ev}_0^{[0,1]})_* \langle \Phi_t \rangle_* = (\text{ev}_1^{[0,1]})_* \langle \Phi_t \rangle_* = \langle \psi_t \rangle_* .$$

□

Lemma 6.5. *Let A and B be R^* -algebras and let $\phi : A \rightarrow B$ be a $*$ -homomorphism. Then $\phi_* = \langle \phi \rangle_* : K_*(A) \rightarrow K_*(B)$.*

Proof. Note that $\widetilde{\langle \phi \rangle} : A \rightarrow \beta B$ which maps a to the constant function $\phi(a)$ is a $*$ -homomorphism and $\overline{\langle \phi \rangle} = \rho_B \circ \widetilde{\langle \phi \rangle}$. Therefore,

$$\langle \phi \rangle_* = (\text{ev}_1^{[0,1]})_*(\rho_B)_*^{-1} \overline{\langle \phi \rangle}_* = (\text{ev}_1^{[0,1]})_* \widetilde{\langle \phi \rangle}_* = \phi_*.$$

□

Lemma 6.6. *Suppose $\phi : B \rightarrow C$ is a $*$ -homomorphism and $\langle \psi_t \rangle : A \rightarrow B$ is an asymptotic morphism. Then $\langle \phi \circ \psi_t \rangle_* = \phi_* \circ \langle \psi_t \rangle_*$.*

Proof. Define $\eta_\phi : \beta B \rightarrow \beta C$ by $\eta_\phi(f)(t) = \phi(f(t))$. Then η_ϕ is a $*$ -homomorphism such that $\eta_\phi(\beta_0 B) \subseteq \beta_0 C$. Thus, there exists a $*$ -homomorphism $\overline{\eta}_\phi : \alpha B \rightarrow \alpha C$ such that

$$\overline{\eta}_\phi \circ \rho_B = \rho_C \circ \eta_\phi \quad \text{and} \quad \overline{\eta}_\phi \circ \overline{\langle \psi_t \rangle} = \overline{\langle \phi \circ \psi_t \rangle}.$$

Hence,

$$\begin{aligned} \langle \phi \circ \psi_t \rangle_* &= (\text{ev}_1^{[1,\infty)})_*(\rho_C)_*^{-1} \overline{\langle \phi \circ \psi_t \rangle}_* \\ &= (\text{ev}_1^{[1,\infty)})_*(\rho_C)_*^{-1} (\overline{\eta}_\phi)_* \overline{\langle \psi_t \rangle}_* \\ &= (\text{ev}_1^{[1,\infty)})_*(\eta_\phi)_*(\rho_B)_*^{-1} \overline{\langle \psi_t \rangle}_* \\ &= \phi_* (\text{ev}_1^{[1,\infty)})_*(\rho_B)_*^{-1} \overline{\langle \psi_t \rangle}_* \\ &= \phi_* \circ \langle \psi_t \rangle_*. \end{aligned}$$

□

As an aside, we note that it is possible to prove a more general statement: the formula $\langle \phi_t \circ \psi_t \rangle_* = \langle \phi_t \rangle_* \circ \langle \psi_t \rangle_*$ holds for asymptotic morphisms $\langle \phi_t \rangle : B \rightarrow C$ and $\langle \psi_t \rangle : A \rightarrow B$.

Theorem 6.7. *There exists non-trivial CRT-homomorphisms of degree 0 of the following forms:*

- (1) $K^{CRT}(S^d \mathbb{R}) \rightarrow K^{CRT}(\mathbb{R})$ where $d > 0$ and $d \equiv 0, 4, 6, 7 \pmod{8}$
- (2) $K^{CRT}(S^d \mathbb{R}) \rightarrow K^{CRT}(\mathbb{H})$ where $d > 0$ and $d \equiv 0, 2, 3, 4 \pmod{8}$.

Furthermore, if $n \in \mathbb{N}$ does not satisfy the specified condition in each case, then there does not exist a corresponding non-trivial CRT-homomorphism.

Proof. The CRT-module $K^{CRT}(S^d \mathbb{R})$ is a free CRT-module with a single generator in the real part in degree $-d$. Hence there exists a non-trivial CRT-module homomorphism $K^{CRT}(S^d \mathbb{R}) \rightarrow M$ if and only if $M_{\mathcal{O}}^{-d} \neq 0$. Now $KO_*(\mathbb{R})$ is non-zero in and only in degrees $0, 1, 2, 4 \pmod{8}$; and $KO_*(\mathbb{H})$ is non-zero in and only in degrees $0, 4, 5, 6 \pmod{8}$. Thus parts (1) and (2) follow as well as the converse statements. □

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. For $d \geq 1$, where d is not in one of the required congruence classes, Theorem 6.7 implies that no asymptotic morphism can exist that induces a non-trivial homomorphism on K -theory.

Now suppose that $d \geq 1$ and suppose that $A = S^d \mathbb{R}$ and $B = \mathbb{R}$ or $B = \mathbb{H}$ are algebras such that the form $A \rightarrow B$ matches the conditions of the statement of Theorem 6.1. By Theorem 6.7, there is a non-zero homomorphism $K^{CRT}(A) \rightarrow K^{CRT}(B)$ and by the Universal Coefficient Theorem for real C^* -algebras (Theorem 1.1 of [3]), this CRT-module homomorphism is induced by a nonzero element $\xi \in \text{KK}(A, B)$.

By Theorem 4.4, there exists an extension $\mathfrak{e} \in \text{Ext}(S^{d-1}\mathbb{R}, B)^{-1}$ such that $\Lambda_{S^{d-1}\mathbb{R}, B}[\mathfrak{e}] = \xi$. Since $[\lambda_{\mathfrak{e}}][\gamma_{\mathfrak{e}}]^{-1} = \xi$ this implies that $[\lambda_{\mathfrak{e}}] \neq 0$ where $[\lambda_{\mathfrak{e}}] \in \text{KK}(A, C_{\pi})$. Furthermore, the homomorphism of \mathcal{CRT} -modules $(\lambda_{\mathfrak{e}})_*: K^{CRT}(A) \rightarrow K^{CRT}(C_{\pi})$ is nonzero. This is because $K^{CRT}(A)$ is a free \mathcal{CRT} -module so the Universal Coefficient Theorem implies that

$$\text{KK}(A, C_{\pi}) \rightarrow \text{Hom}_{CRT}(K^{CRT}(A), K^{CRT}(C_{\pi}))$$

is an isomorphism. Even more, this implies that the homomorphism on K -theory $(\lambda_{\mathfrak{e}})_*: K_*(A) \rightarrow K_*(C_{\pi})$ is nonzero, by Theorem 4 of [4].

Let $\langle \phi_t^{\mathfrak{e}} \rangle : S^d\mathbb{R} \rightarrow B$ be the asymptotic morphism associated to \mathfrak{e} . Then by Proposition 6.2, $\langle \gamma_{\mathfrak{e}} \circ \phi_t^{\mathfrak{e}} \rangle$ is homotopic to $\langle \lambda_{\mathfrak{e}} \rangle$. Therefore, by Lemmas 6.4, 6.5, and 6.6; we have

$$(\gamma_{\mathfrak{e}})_* \circ \langle \phi_t^{\mathfrak{e}} \rangle_* = (\lambda_{\mathfrak{e}})_*.$$

Since $(\lambda_{\mathfrak{e}})_* \neq 0$, it follows that $\langle \phi_t^{\mathfrak{e}} \rangle_* \neq 0$. □

7. ALMOST COMMUTING MATRICES

We can use all this machinery to find novel examples of almost commuting real symmetric matrices. Our approach is to use commutative R^* -algebras and create asymptotic morphisms out of these. On the one hand these carry K -theory data that can distinguish them from actual $*$ -homomorphisms. On the other, in the image the relations that make the R^* -algebra commutative get turned into the property of being almost commuting.

Let \mathbb{F} denote either \mathbb{R} or \mathbb{C} or \mathbb{H} . We generalize a question of Halmos [12] to ask about d almost commuting self-adjoint matrices over \mathbb{F} .

Problem 7.1. For all $\epsilon > 0$, does there exist $\delta > 0$ so that, for all n , given d self-adjoint contractions H_r in $\mathbf{M}_n(\mathbb{F})$ such that

$$\|[H_r, H_s]\| \leq \delta,$$

there exist d self-adjoint contractions K_r with $\|K_r - H_r\| \leq \epsilon$ and

$$[K_r, K_s] = 0?$$

Lin [23] showed that in the complex case the answer is “yes” for $d = 2$ while it was known much earlier [35] that the answer is “no” in the complex case for $d = 3$.

For the quaternionic case, the result is the same: yes for $d = 2$ [27] and no for $d = 3$ [14].

These leaves the real case, arguably the most important case. We know the answer is “yes” for $d = 2$ ([27]). We will show that the answer is “no” for $d = 5$, leaving open the cases $d = 3, 4$. The proof techniques used for a negative result for $d = 3$ in the complex and quaternionic cases rely on the fact that $K_{-2}(\mathbb{H}) \neq 0$ and $K_{-2}(\mathbb{C}) \neq 0$ and so will not work for $\mathbb{F} = \mathbb{R}$ since $K_{-2}(\mathbb{R}) = 0$. However, since $K_{-4}(\mathbb{R})$ is nontrivial, we will see that these methods will apply for $d = 5$.

We start by connecting this problem to a problem couched in the theory of R^* -algebras. For any sequence B_n of R^* -algebras, let π be the quotient map from the product $\prod_{n=1}^{\infty} B_n$ to its quotient by the sum $\prod_{n=1}^{\infty} B_n / \bigoplus_{n=1}^{\infty} B_n$.

Problem 7.2. Does every $*$ -homomorphism of the form

$$\psi: S^{d-1}\mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

where $\{m(n)\}_{n=1}^{\infty}$ is a sequence of integers, lift to a $*$ -homomorphism

$$\tilde{\psi}: S^{d-1}\mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

such that $\psi = \pi \circ \tilde{\psi}$?

Theorem 7.3. *For a fixed positive integer d and division algebra \mathbb{F} , if the answer to Problem 7.1 is “yes” then the answer to Problem 7.2 is also “yes”.*

We prove Theorem 7.3 below, following two lemmas. Then, the rest of the section is devoted to showing that the answer to Problem 7.2 is “no” when $d = 5$, using a K -theoretic obstruction. We start with a lemma stating the universal properties of $C(S^{d-1}, \mathbb{R})$. This is a standard result that can be proven using the techniques of Chapter 3 of [24].

Lemma 7.4. *Let A be an R^* -algebra. If h_1, \dots, h_d are commuting self-adjoint contractions in A that satisfy $\sum_{j=1}^d h_j = 1$, then there is a unique $*$ -homomorphism $\psi: C(S^{d-1}, \mathbb{R}) \rightarrow A$ sending the j th coordinate function f_j to h_j .*

Lemma 7.5. *For each $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds. Let k_1, \dots, k_d be commuting self-adjoint elements in a unital R^* -algebra A that satisfy $\|\sum_{i=1}^d k_i^2 - 1\| < \delta$. Then there exist commuting self-adjoint elements k'_1, \dots, k'_d in A such that $\|k'_i - k_i\| < \varepsilon$ and $\sum_{i=1}^d (k'_i)^2 = 1$.*

Proof. Assume that $0 < \delta < .5$ and that the elements k_i satisfy $\|\sum_{i=1}^d k_i^2 - 1\| < \delta$. Since the elements k_i commute, we can treat them as real-valued functions in $C(X, \mathbb{R})$ for some compact space X . Let $r(x) = \sqrt{1/\sum_{i=1}^d k_i^2(x)}$. Then $k'_i(x) = r(x)k_i(x)$ satisfies $\sum_{i=1}^d (k'_i)^2 = 1$. A short calculation shows that $\|k_i - k'_i\| = \left\|1 - \sqrt{\frac{1}{\sum_{i=1}^d k_i^2}}\right\|$. Thus for any $\varepsilon > 0$, we can find a δ small enough to ensure that $\|k_i - k'_i\| < \varepsilon$. \square

Proof of Theorem 7.3. Suppose that the answer to Problem 7.1 is “yes” for some d and \mathbb{F} , and let

$$\psi: C(S^{d-1}, \mathbb{F}) \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

be a $*$ -homomorphism. Then taking the images of the coordinate functions in $C(S^{d-1}, \mathbb{F})$ we find that there exist sequences of matrices $H_{in} \in \mathbf{M}_{m(n)}(\mathbb{F})$ ($i \in \{1, \dots, d\}$, $n \in \mathbb{N}$) that are asymptotically (as $n \rightarrow \infty$) self-adjoint contractions, that satisfy $\sum_{i=1}^d H_{in}^2 = 1$ asymptotically, and such that H_{in} and H_{jn} asymptotically commute for each $i, j \in \{1, \dots, d\}$. In fact, we may assume that each H_{in} is exactly a self-adjoint contraction. This is achieved by replacing H_{in} by $\frac{1}{2}(H_{in} + H_{in}^*)$ and then by $f(H_{in})$ where $f \in C(\mathbb{R}, \mathbb{R})$ is defined by

$$f(x) = \begin{cases} -1 & x \leq -1 \\ x & -1 < x < 1 \\ 1 & x \geq 1. \end{cases}$$

We may also assume without loss of generality that $\|\sum_{i=1}^d h_{in}^2 - 1\| \leq .5$ for all n .

By our hypothesis, there exists self-conjugate contractions $K_{in} \in \mathbf{M}_{m(n)}(\mathbb{F})$ that exactly commute and satisfy $\lim_{n \rightarrow \infty} (H_{in} - K_{in}) = 0$ for each i . By Lemma 7.5, there exists matrices $L_{in} \in \mathbf{M}_{m(n)}(\mathbb{F})$ such that $\lim_{n \rightarrow \infty} (K_{in} - L_{in}) = 0$ for all i and $\sum_{i=1}^d L_{in}^2 = 1$ for each n .

Then by Lemma 7.4, there exist $*$ -homomorphisms $\psi_n: C(S^{d-1}, \mathbb{R}) \rightarrow \mathbf{M}_{m(n)}(\mathbb{F})$ that map the d coordinate functions to L_{in} . These maps form a $*$ -homomorphism

$$\tilde{\psi}: C(S^{d-1}, \mathbb{R}) \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}).$$

Since $\lim_{n \rightarrow \infty} \|L_{in} - H_{in}\| = 0$ it follows by the uniqueness statement of Lemma 7.4 that $\pi \circ \tilde{\psi} = \psi$. \square

Lemma 7.6. *Suppose \mathbb{F} is a division algebra and $d \geq 1$. Any $*$ -homomorphism*

$$S^k \mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

is homotopic to zero.

Proof. It suffices to show that any $*$ -homomorphism $\varphi : S^d \mathbb{R} \rightarrow \mathbf{M}_{m(n)}(\mathbb{F})$ is homotopic to the zero map. As the codomain is finite dimensional, this φ must factor through $C(X, \mathbb{R})$ where X is finite. Thus φ is a sum of point evaluations, each of which is homotopic to 0. \square

Let Θ be the map from $K_*(\prod_{n=1}^{\infty} B_i)$ to $\prod_{n=1}^{\infty} K_*(B_i)$ induced by the collection of projection maps

$$\pi_i : \prod_{n=1}^{\infty} B_i \rightarrow B_i .$$

In [11], there is a thorough analysis of the circumstances under which Θ is surjective and injective on K_0 and K_1 for C^* -algebras B_i . The following lemma states that the situation for R^* -algebras partially reduces to that for C^* -algebras.

Lemma 7.7. *Let $\{B_i\}_{i \in \mathbb{N}}$ be a sequence of R^* -algebras and let $(B_i)_{\mathbb{C}}$ be their respective complexifications. Then the map*

$$\Theta^{CRT} : K^{CRT} \left(\prod_{n=1}^{\infty} B_i \right) \rightarrow \prod_{n=1}^{\infty} K^{CRT}(B_i)$$

is an isomorphism if and only if

$$\Theta : K_* \left(\prod_{n=1}^{\infty} (B_i)_{\mathbb{C}} \right) \rightarrow \prod_{n=1}^{\infty} K_*((B_i)_{\mathbb{C}})$$

is an isomorphism.

Proof. Since K^{CRT} is a CRT -module homomorphism, this follows immediately from the results of Section 2.3 of [6]. \square

We shall say that a family of R^* -algebras $\{B_i\}_{i \in \mathbb{N}}$ has *property Θ* if Θ^{CRT} is an isomorphism. We shall say further that an R^* -algebra B has *property Θ* if $\{B_i\}_{i \in \mathbb{N}}$ has property Θ where $B_i = B$ for all $i \in \mathbb{N}$. In particular $\mathbb{F} \otimes \mathcal{K}_{\mathbb{R}} \cong \mathcal{K}_{\mathbb{F}}$ has property Θ .

We note that if B has property Θ , then the $*$ -homomorphism

$$\Delta : B \rightarrow \prod_{n=1}^{\infty} B$$

and the resulting map

$$\hat{\Delta} : B \rightarrow \prod_{n=1}^{\infty} B \Big/ \bigoplus_{n=1}^{\infty} B$$

into the quotient are both injective on K -theory and united K -theory.

Recall that we have $\beta B = C_b([1, \infty), B)$ and $\alpha B = C_b([1, \infty), B)/C_0([1, \infty), B)$. We can evaluate an element of βB at the natural numbers, and get a surjection

$$\delta_{\mathbb{N}} : \beta B \rightarrow \prod_{n=1}^{\infty} B ;$$

which passes to a $*$ -homomorphism

$$\hat{\delta}_{\mathbb{N}}: \alpha B \rightarrow \prod_{n=1}^{\infty} B \Big/ \bigoplus_{n=1}^{\infty} B.$$

Lemma 7.8. *Suppose A is a separable R^* -algebra and B is a separable R^* -algebra with property Θ . If*

$$\langle \phi_t \rangle : A \rightarrow B$$

is an asymptotic morphism, then there is a $$ -homomorphism*

$$\psi : A \rightarrow \prod_{n=1}^{\infty} B \Big/ \bigoplus_{n=1}^{\infty} B$$

so that on K -theory

$$\psi_* = \hat{\Delta}_* \circ \langle \phi_t \rangle_*.$$

In particular, if $\langle \phi_t \rangle_$ is non-zero, then so is ψ_* .*

Proof. Define $\psi = \hat{\delta}_{\mathbb{N}} \circ \overline{\langle \phi_t \rangle}$. Consider the following diagram. The square commutes exactly. The triangle on the top commutes at the level of K -theory, as can be seen using the fact that δ_1 is homotopic to δ_k for any $k \in \mathbb{N}$ and identifying the K -theory of the product with the product of the K -theory.

$$\begin{array}{ccccc} & & & B & \\ & & \nearrow \delta_1 & \downarrow \Delta & \\ \beta B & \xrightarrow{\delta_{\mathbb{N}}} & \prod_{n=1}^{\infty} B & & \\ \downarrow \rho_B & & \downarrow \pi & & \\ A \xrightarrow{\overline{\langle \phi_t \rangle}} \alpha B & \xrightarrow{\hat{\delta}_{\mathbb{N}}} & \prod_{n=1}^{\infty} B \Big/ \bigoplus_{n=1}^{\infty} B & & \end{array}$$

Using the relation $\langle \phi_t \rangle_* = \delta_1 \circ (\rho_B)_*^{-1} \circ \overline{\langle \phi_t \rangle}_*$, an easy calculation shows that $\hat{\Delta}_* \circ \langle \phi_t \rangle_* = \psi_*$ \square

Lemma 7.9. *Suppose B is a separable R^* -algebra, and A is a finitely generated R^* -subalgebra of*

$$A \subseteq \prod_{n=1}^{\infty} (B \otimes \mathcal{K}_{\mathbb{R}}) \Big/ \bigoplus_{n=1}^{\infty} (B \otimes \mathcal{K}_{\mathbb{R}}).$$

Then there is a sequence $m(1) < m(2) < \dots$ on natural numbers so that

$$A \subseteq \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(B) \Big/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(B).$$

Proof. Given a single element $a \in A$, write

$$a = (a_1, a_2, \dots) + \bigoplus (B \otimes \mathcal{K}_{\mathbb{R}})$$

where $a_i \in B \otimes \mathcal{K}_{\mathbb{R}}$. We can choose an increasing sequence p_1, p_2, \dots of standard projections in $1 \otimes \mathcal{K}_{\mathbb{R}}$ (with 1 in \tilde{B} if needed) so that

$$\|p_n b_n p_n - b_n\| \leq \frac{1}{n}$$

and so

$$(b_1, b_2, \dots) + \bigoplus (B \otimes \mathcal{K}_{\mathbb{R}}) = (p_1 b_1 p_1, p_2 b_2 p_2, \dots) + \bigoplus (B \otimes \mathcal{K}_{\mathbb{R}}).$$

More generally, for a finite set of elements in A , we can use a single sequence of projections as above to show that

$$A \subseteq \prod_{n=1}^{\infty} p_n (B \otimes \mathcal{K}_{\mathbb{R}}) p_n \bigg/ \bigoplus_{n=1}^{\infty} p_n (B \otimes \mathcal{K}_{\mathbb{R}}) p_n.$$

□

We now put these results together for our main theorem.

Theorem 7.10. *Suppose that $d \in \mathbb{N}$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$ satisfies the hypotheses of Theorem 6.1. Then there is a sequence of integers $m(1), m(2), \dots$ and a unital $*$ -homomorphism*

$$\varphi : C(S^d, \mathbb{R}) \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

that cannot be lifted to a unital $*$ -homomorphism

$$\psi : C(S^d, \mathbb{R}) \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}).$$

Proof. Let d be as above. By Theorem 6.1, there exists an asymptotic morphism

$$\langle \phi_t \rangle : S^d \mathbb{R} \rightarrow \mathcal{K}_{\mathbb{F}}$$

that induces a non-zero map on K -theory. Then by Lemma 7.8, we obtain a $*$ -homomorphism of the form

$$\phi' : S^d \mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}} \bigg/ \bigoplus_{n=1}^{\infty} \mathcal{K}_{\mathbb{F}}$$

that is non-zero on K -theory. By Lemma 7.9, we have that this $*$ -homomorphism factors through a $*$ -homomorphism of the form

$$\phi : S^d \mathbb{R} \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}).$$

Since $\phi' = \iota \circ \phi$, where ι is the inclusion

$$\iota : \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(B) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(B) \rightarrow \prod_{n=1}^{\infty} B \otimes \mathcal{K}_{\mathbb{F}} \bigg/ \bigoplus_{n=1}^{\infty} B \otimes \mathcal{K}_{\mathbb{F}},$$

it follows that ϕ_* is also nonzero. Now ϕ cannot be lifted to a $*$ -homomorphism ψ with values in $\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$ since such a lift would have to be non-zero on K -theory contradicting Lemma 7.6.

Finally, extend unitally to form a $*$ -homomorphism

$$\phi : C(S^d, \mathbb{R}) \rightarrow \prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F}) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{F})$$

that similarly cannot be lifted. □

Corollary 7.11. *For $d = 5$, $\mathbb{F} = \mathbb{R}$ and for $d = 3$, $\mathbb{F} = \mathbb{H}$; the answer to Problem 7.2, and hence also to Problem 7.1, is “no”.*

The result above for $d = 3$, $\mathbb{F} = \mathbb{H}$, replicates results from [27]. Considering this case more closely, we can clarify the K -theory behind the invariant for 3D topological insulators considered in [14]. We consider \mathbb{H} as sitting inside its complexification, so is to be regarded as the set of

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$$

where a and b are complex numbers. More generally, we identify $\mathbf{M}_N(\mathbb{H})$ with $2N$ -by- $2N$ complex matrices of the form

$$\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}.$$

See [26].

The first several homotopy groups for $\mathrm{Sp}(n)$ stabilize early. The connection with K -theory is that $K_1(C(S^3, \mathbb{H}))$ is given by homotopy classes of maps from the three sphere into the symplectic unitary matrices of various sizes. Let x_1, \dots, x_4 be the coordinate functions in \mathbb{R}^4 restricted to the unit sphere S^3 . From [36, §10.6] we learn that

$$\begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \in C(S^3, \mathbb{H})$$

is a generator of

$$\pi_3(\mathrm{Sp}(1))$$

and so gives us a generator for

$$K_{-3}(C(S^3, \mathbb{R})) \cong K_1(C(S^3, \mathbb{H})) \cong \mathbb{Z}/2.$$

The K_1 group of

$$\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{R}) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{R})$$

is easy enough to work out, since it is entirely governed by the determinants of real orthogonal matrices. It is a subgroup of

$$\prod_{n=1}^{\infty} \mathbb{Z}/2 \bigg/ \bigoplus_{n=1}^{\infty} \mathbb{Z}/2.$$

When $d = 3$, the induced map φ_* on K -theory is such that the generator of

$$K_{-2}(C(S^3, \mathbb{R}))$$

is sent to the class of

$$(\bar{1}, \bar{1}, \dots)$$

in

$$K_{-2} \left(\prod_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{H}) \bigg/ \bigoplus_{n=1}^{\infty} \mathbf{M}_{m(n)}(\mathbb{H}) \right) = \prod \mathbb{Z}_2 \bigg/ \bigoplus \mathbb{Z}_2.$$

This means that the \mathbb{Z}_2 -index [14] can be non-trivial for arbitrarily small commutators. Now we switch to considering $\mathbf{M}_N(\mathbb{H})$ as a real part of $\mathbf{M}_{2N}(\mathbb{C})$ with respect to the dual operation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\sharp} = \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix}.$$

That is, there are self-dual self-adjoint matrices H_1, \dots, H_4 with

$$\|[H_r, H_s]\| \leq \delta, \quad \left\| \sum_{r=1}^4 H_r^2 - I \right\| \leq \delta$$

for any small δ , so that

$$\det \begin{bmatrix} H_1 + iH_2 & -H_3 + iH_4 \\ H_3 + iH_4 & H_1 - iH_2 \end{bmatrix} < 0.$$

This is evidence that the phenomena of 3D systems with topologically protected states as witnessed in [14] would persist in larger system size.

8. THE TEN-FOLD WAY

Some of the motivation of this work is to explain index studies of topological insulators. The symmetries present in topological insulators are essential in their behavior. Physicists think of time-reversal (TR) symmetry in terms of commuting relations with anti-unitary operators, but it is equivalent to think in terms of anti-multiplicative involutions.

For single particle systems, this picture suffices. There can be no TR symmetry, where the relevant algebra for a finite model is the C^* -algebra $\mathbf{M}_N(\mathbb{C})$. When TR symmetry is present, the TR symmetry comes in two forms, and leading to either the transpose or the dual operation. We are looking at Dyson's "three-fold" way, or real, complex or quaternionic matrices.

In more complex systems, one needs to track additional symmetries, in particular particle-hole symmetry. The picture here, in terms of algebras, is that there are two anti-multiplicative involutions τ and γ and the Hamiltonian $H \in \mathbf{M}_N(\mathbb{C})$ now satisfies

$$H^* = H, \quad H^\tau = H, \quad H^\gamma = -H.$$

There are more general symmetry groups to consider, acting by unitary and anti-unitary operators. See [13]. For now, we consider just the symmetries beyond being self-adjoint.

In physics, it is expected that these two " τ -operations" will commute. As one can have either symmetry non-existent, or equivalent to the transpose, or equivalent to the dual operation, it seems that we will get nine symmetry classes. Instead, we find ten as the combined symmetry of τ followed by γ can be preserved, or broken, when each is individually broken. So condensed matter physics is now investigated in terms of the ten-fold way. See [18] for a nice introduction to the matrix classes that arise.

It is easy to check that when τ and γ commute,

$$\alpha(a) = (a^\tau)^\gamma$$

defines an order-two automorphism. If we are to follow the ten-fold way, we need to be working with graded real C^* -algebras.

It was in the context of graded real C^* -algebras KK-theory was introduced. In work on the standard model [8], Connes considered real structures on spectral triples. It seems likely that progress in graded real noncommutative geometry will have application in the study of topological insulators.

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